TIMING OF INVESTMENT UNDER TECHNOLOGICAL AND REVENUE RELATED UNCERTAINTIES

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Abstract: We consider the effects of technological and revenue related uncertainties on the timing of irreversible investment. The distinction between the two types of uncertainty is first characterized within the optimal stopping framework. Then, a specific model motivated by investment in wind power production is presented. Technological progress is modeled as a Poisson arrival process that reduces the cost of investment, and revenue uncertainty is modeled as a geometric Brownian motion process. We show that in the absence of revenue uncertainty the technological uncertainty does not affect the optimal investment rule. However, when combined with revenue uncertainty, increased technological uncertainty makes investment less attractive relative to waiting.

Timing of investment under technological and revenue related uncertainties[*]

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JEL classification: D81, G31, O31, Q40

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1 Introduction

Technological progress affects greatly the development of industries. Energy sector may serve as an example of this. For instance, in the 1980's the technology imported from materials science and the space programme made turbines much more efficient than before. This made it possible to build smaller and cheaper natural gas burning generation units, which led to a dramatic reduction in the optimal power plant size (Hunt and Shuttleworth, 1996). Due to this and a number of other technological innovations the cost of generating a kWh of electricity has reduced considerably during the last decades.

A current trend in the energy sector is the increasing use of renewable technologies such as wind power and biomass. There is a political support for this development motivated mostly by environmental reasons and the desire to reduce dependence on exports. The European Union, for example, has set an explicit target of increasing the share of renewables in energy consumption to 12 % by 2010 (European Commission, 1997). However, at current technologies, renewable sources of energy are not competitive with fossil fuels without substantial subsidies (e.g. European Commission, 2001). Therefore, how fast and to what extent renewable technologies are able to take over the energy supply depends much on the progress in these technologies in the forthcoming years and decades.

The objective of this paper is to provide a deeper look at the influence of technological progress on investment. We approach the issue by looking at the optimal behavior of a value maximizing firm, which faces an investment opportunity subject to exogenous technological progress. Different uncertainties play a key role in this context. For example, with capital-intensive renewable energy projects with long payback times, the foremost uncertainty concerns the revenues that the plant will generate after the up-front investment cost has been sunk. For a power plant, this is mostly due to the uncertain development of electricity price.[1] Such revenue uncertainty should certainly be taken into account by an investor facing the decision of when, if ever, to carry out a particular project.

However, technological progress itself contains another source of uncertainty that has attained less attention. Even if the subsequent technological improvements would not affect the values of production facilities that already exist, an investor deciding whether to carry out an investment project now or perhaps later must take into consideration the fact that postponing the investment may allow the accomplishment of the project later with an improved technology. For projects with pay-back horizons extending over many decades, such considerations may be very important. The history of wind power production supports this view with examples of cost reducing technological innovations.[2] As reported in Krohn (2002): "The economics of wind energy has improved tremendously during the past 15 years tumbling with a factor of five."

There is a conceptual difference between uncertainty in technological progress and revenue. Namely, a characteristic property of technological progress is that it moves in one direction only. In other words, innovations can only improve the best-available technology, not worsen it. Therefore, when pointing to uncertainty in technological progress, we refer to the speed at which the technology improves, not the direction in which it moves. This is in contrast with the revenue uncertainty, where the income stream is typically subject to both up- and down-ward shocks.

In this paper we consider the effects of these two types of uncertainties on the timing of investment. The analysis is based on the theory of the irreversible investment under uncertainty. This approach, also referred to as the real options approach, considers problems where a firm should choose the optimal timing of investment when the decision can not be reversed and the value of the project evolves stochastically. Important contributions to the theory include, e.g., McDonald and Siegel (1986) and Pindyck (1988), while a thorough review of the techniques and literature is given in the book of Dixit and Pindyck (1994).[3]

When classifying the existing real options literature according to the above mentioned two types of uncertainties, almost all the papers fall to the category of revenue uncertainty. Exceptions in the other category are Grenadier and Weiss (1997) and Farzin et al. (1998). An earlier related study is Balcer and Lippman (1984). However, these papers consider only technological uncertainty, whereas we specify the distinction between the two types of uncertainties and show how they act together.

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[1] Other sources of uncertainty are also possible depending on the institutional setting and the production technology. For example, the market for green certificates has been proposed in Europe, where power producers using renewable technologies receive an additional income from selling the certificates. See, e.g., Amundsen and Mortensen (2001) or Jensen and Skytte (2002) for economic analysis of such markets.

[2] See, e.g., the history of wind turbines at the www-site of the Danish Wind Industry Association: www.windpower.org. Major technological breakthroughs are reported: "The 55 kW generation of wind turbines which were developed in 1980 - 1981 became the industrial and technological breakthrough for modern wind turbines. The cost per kilowatt hour (kWh) of electricity dropped by about 50 percent with the appearance of this generation of wind turbines".

[3] Applications in energy investments include Paddock et al. (1988), Martzoukos and Teplitz-Sembitzky (1992), Pindyck (1993), Brekke and Schieldrop (1999), and Venatsanos et al. (2002). Other related applications are, e.g., Brennan and Schwartz (1985) and Lumley and Zervos (2001), who consider natural resource investments.

2

Also Alvarez and Stenbacka (2001, 2002) consider technological uncertainty alongside revenue uncertainty, but in their models this concerns only events that occur after the irreversible investment has been undertaken. We are interested in exogenous technological progress, which the investor observes already before undertaking the project.

Technically, the problem of choosing the timing of irreversible investment is an optimal stopping problem. To emphasize this, we start the paper with a general model of investment in that framework. This allows us to characterize the difference from the investor's point of view between the uncertainty in revenue stream and in technological progress. We show that in the latter case, which we characterize by the property that the state variables are non-decreasing stochastic processes, the solution gets a very intuitive form. Namely, the investment is carried out at a moment when the opportunity cost of delaying the project equals the expected change in its net present value. This means that the optimal decision of whether to invest now or later depends only on the expected path of the project value, not on its probability distribution. This rule does not, however, work with revenue uncertainty, because then the stochastic state variables fluctuate both up and down, and the investor must take into account the value of flexibility in being non-committed to the investment. This option value damps investment, and is created by the fact that conditions may later turn worse thus making the investor better off by having held back from investing.

We then proceed to present a more specific model of an investment opportunity where both technological and revenue related uncertainties are present. The technological progress is modeled as a Poisson arrival process, where innovations that reduce the cost of investment arrive at random times. The revenue stream that the investment would generate follows a geometric Brownian motion. The investor observes these two processes, and must decide when the investment cost is low enough and revenue stream is high enough to carry out the investment. The model is motivated by wind power investments, in which case the investment opportunity is represented by a given site suitable for wind power production, and the revenue stream is represented by the electricity price. The site can be seen as a natural resource owned by an investor who wants to optimize its utilization in order to maximize its value. To see the analogy to options theory, note that the site is a real-option contingent on the two underlying stochastic factors, giving the owner an opportunity but no obligation to develop it for wind power production.

The uncertainty in the technological progress gives rise to some interesting findings. In the absence of revenue uncertainty (i.e. when the volatility of the revenue process is set to zero), the technological uncertainty as such does not matter. The investor can act as if the actual stochastic process for the investment cost were replaced by its expected path. However, when the revenue uncertainty is added in the model, the technological uncertainty starts to matter as well. Namely, keeping the expected path of the investment cost fixed, the higher the uncertainty in the process, the more reluctant the investor is to invest. It is perhaps against common intuition that the effect of technological uncertainty depends crucially on whether the revenue stream is deterministic or stochastic.

It is worth emphasizing that even if motivated by wind power investments, the relevance of our findings is not restricted to energy sector. One can think of applications in many different areas with similar characteristics. A possible application could be, for example, the development of a new product, where the developer must decide when the product and the market conditions are good enough for market introduction. Moreover, instead of investing in a new production unit (as in the case of wind power investment), one can also think of the adoption of a new technology

3

to replace an older one.[4] For example, think of an organization planning to update to the latest computer operating system, a farm considering to switch to a new cropping technology, or a factory considering to switch to a more efficient production line. In all of these cases the decision maker must take into account the fact that delaying the investment may allow one to later switch to an improved version. Moreover, it is not difficult to imagine uncertainties in the arrival of technological improvements, or in the value that the switching would create.

The paper is structured as follows. In section 2 we review a general model of investment in the optimal stopping framework. The role of this section is to characterize the distinction between the uncertainties in technological progress and in revenue, and to enable one to explain the results of the later sections. We illustrate the main message of the section through a series of examples given in Appendix A. In section 3, we present the specific model of investment where both technological and revenue uncertainties are present. The model can not be solved in closed form, but in section 4 we solve it in three different special cases, which together capture the main insights of the paper. Section 5 concludes.

2 Investment as an optimal stopping problem

In the literature of irreversible investment under uncertainty, a typical investment problem is characterized by the following three features. First, the value of the asset obtained with the investment is a stochastic process, second, the investment decision is irreversible, and third, the investor is free to choose the timing of investment. In such a setting the problem of choosing the timing of investment is technically an optimal stopping problem. In this section, we review the basic ideas of such a problem. Using this formulation we then characterize the distinction between the problem where uncertainty concerns the revenue stream and the problem where uncertainty concerns the arrival of technological innovations. To keep the key ideas easily readable, we maintain the lowest possible level of technical formality. See, e.g., Øksendal (2000) or Karatzas and Shreve (1998), for formal treatments of optimal stopping problems.

2.1 General problem

We consider an investor who has an opportunity to make a single irreversible investment. The time is continuous and infinite, and we denote by $t \in [0, \infty)$ the time index. The value of the investment is affected by some stochastic factors. More precisely, we have n real-valued stochastic processes denoted $\{X_t^i\}$, $i = 1, ..., n$. We denote by $X_t = [X_t^1, ..., X_t^n] \in R^n$ the vector containing the values of the processes at time t. We assume that the processes are independent[5] Markov processes, and thus we say that X_t represents the *state* of the model. By $\{X_t\}$ we refer to the process with values in R^n that contains processes $\{X_t^i\}$, $i = 1, ..., n$, and by $X = [X^1, ..., X^n]$ simply to the value of the state without referring to the calendar time.

Typically, the stochastic factors are variables that affect the revenue stream that is obtained once the investment has been undertaken, for example output prices or

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[4] There is an extensive theoretical literature dealing with different aspects of the problem. Many papers focus on the strategic interaction (see, e.g., Hoppe, 2000, or Rahman and Loulou, 2001, for recent contributions, or Reingamum, 1981, or Fudenberg and Tirole, 1985, for well known older papers), but there are also papers that focus on uncertainty, see, e.g., Balcer and Lippman (1984), or Farzin et al. (1998).

[5] The results do not rest on the assumption that the processes are independent. In reality, prices of commodities are typically correlated. The assumption is made because it simplifies the exposition by reducing the need for formalism.

demand. We use the term *revenue uncertainty* to refer to such uncertainty.[6] The book by Dixit and Pindyck (1994) reviews extensively this kind of models. They are mostly in one dimension, and the stochastic process used in most of the cases is the geometric Brownian motion.

However, technological progress may also represent a relevant stochastic factor. Improving technology may, for example, lower the cost of investment thus increasing the net value of investment, as we will assume in section 3. The important property of technological progress is that it typically moves in one direction only, whereas revenue uncertainty is due to both positive and negative random fluctuations. We will consider the effect of this refining property characteristic to technological uncertainty in section 2.2.

Assume that the value of the investment project at time t depends on the state X_t, but not on the calendar time. Thus, if the investment is undertaken at time t, the net present value of the project is $V(X_t)$. The investor observes the evolution of the processes $\{X_t^i\}$ and knows exactly the probability laws that they follow. The problem is to choose the timing of investment in such a way that the expected discounted value of the investment is maximized, i.e., to choose the time τ in order to maximize $E\left(e^{-r\tau}V(X_\tau)\right)$. Since it is not possible to anticipate the future, the decision of whether to stop or not at a given time can only depend on the past behavior of the processes $\{X_t^i\}$. Technically, this means that the investment time must be a stopping time.[7] We denote by $F(X_t)$ the value of the investment opportunity given the current state X_t when the timing of investment τ^* is chosen optimally:

$$F(X_t) = E\left(e^{-r(\tau^*-t)}V(X_{\tau^*})\right) = \sup_\tau E\left(e^{-r(\tau-t)}V(X_\tau)\right), \qquad (1)$$

where r is the discount factor[8], E denotes the expectation with respect to the processes $\{X_t^i\}$ when they start from X_t, and the supremum is taken over all stopping times τ for $\{X_t\}$. Obviously, since it is always possible to choose $\tau = t$, it must be that $F(X_t) \geq V(X_t)$.

We have to make certain assumptions on the properties of the stochastic processes and the value of the project to make sure that the problem has a certain nice structure. For the processes $\{X_t^i\}$ we require two properties besides the Markov property. First, we assume that they are *time-homogenous*. This means that the calendar time does not affect the evolution of the process.[9] Second, to have certain regularity we require a *positive serial correlation*, i.e., persistence of uncertainty, meaning roughly that the higher the value of the process is now, the higher the probability that the value is high in the future. For the value of the project we require that V is monotonic and continuous in all components X^i. Without loss of generality, we assume that V is *increasing* in all of its arguments.

All of the assumptions make economically sense. In this paper, the stochastic processes considered are the geometric Brownian motion and the Poisson jump process, which are time-homogenous Markov processes with positive serial correla-

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[6]Of course, we may also have stochastic variables that affect the cost flows that occur after the investment has been undertaken, such as input prices. The uncertainty in negative cash flows has in essence the same effect as uncertainty in positive cash flows. For simplicity, however, we use the term revenue uncertainty throughout the paper.

[7]See, e.g., Øksendal (2000) for a formal definition.

[8]This assumes that the investor is risk neutral. However, the model may also be interpreted so that the stochastic factors are spanned by financial markets and the objective is stated using the equivalent risk-neutral valuation. In that case, the risk-aversity is accounted for by presenting the stochastic processes under an appropriately adjusted probability measure (see, e.g., Dixit and Pindyck, 1994, section 4.3.A).

[9]More precisely, the probability distribution of X_{t+s}^i conditional on $X_t^i = x^i$ is equal to the probability distribution of X_{t+h+s}^i conditional on $X_{t+h}^i = x^i$ for any $t, h, s \in (0, \infty)$ and $x^i \in R$.

tion. The value function that we consider is linear in the state value, and therefore of course monotonic and continuous.

Together these assumptions imply several important things about the structure of the problem. First, the time-homogeneity and Markov properties mean that at a given time t, the decision of whether to stop or wait depends only on the current value of the state X_t, not on the calendar time t or the history of the processes. This implies that the state-space can be divided into two parts, the stopping region where it is optimal to invest and the continuation region where it is optimal to wait. Since V is continuous, the stopping region can be expressed as a closed set $\Omega \subset R^n$. The optimal investment time is a *first-passage time*, i.e., the first time when the state X_t enters Ω. Since the calendar time does not affect the investment decision, we will from here on leave the subscript t out and refer by X to the current state value. Second, because of the persistence of uncertainty and the fact that V is increasing in all components of X, increasing the values of the stochastic processes should always make the investing more tempting. Therefore, the stopping region must be a connected region with the property that if a point $X_1 = \left[X_1^1, ..., X_1^n\right] \in \Omega$, then $X_2 = \left[X_2^1, ..., X_2^n\right] \in \Omega$ whenever $X_2^i \geq X_1^i$ for all $i = 1, ..., n$.[10] The investment occurs at the first moment when the state X crosses the border of Ω, which we denote $\partial\Omega$. The problem is thus to find the optimal $\partial\Omega$.

We have presented the problem in a too general form for presenting a detailed solution method. The standard technique to approach the problem is to use the dynamic programming. This leads to the Bellman function, which in the present case takes the form of the following condition that must hold in the continuation region:

$$rF\left(X\right)dt = E\left(dF\left(X\right)\right), \text{ when } X \notin \Omega, \tag{2}$$

where $dF\left(X\right)$ is the change in the value of F caused by the (stochastic) change in X within an infinitesimal time increment dt. If X follows an Ito process (such as the Brownian motion), then an expression for dF can be derived using Ito's lemma. The intuitive meaning of (2) is that the opportunity cost of holding the investment option through the interval dt, namely $rF\left(X\right)dt$, must be equal to the expected gain in the value of the investment opportunity, namely $E\left(dF\left(X\right)\right)$.

Whenever X is in the stopping region, the rational investor invests without any delay, so the value of the option to invest must be equal to the net present value of the project:

$$F\left(X\right) = V\left(X\right), \text{ when } X \in \Omega. \tag{3}$$

To solve the problem, one should find Ω and a function $F\left(X\right)$ such that (2) and (3) are satisfied. Condition (3) is especially relevant at the boundary $\partial\Omega$ ($\partial\Omega \subset \Omega$, because Ω is closed). Conditions (2) and (3), however, are not normally sufficient for obtaining a unique solution. In many cases, the optimal solution is found by applying an additional condition called the high contact principle or the smooth-pasting condition. The condition says that at the boundary $\partial\Omega$, the first derivatives of F with respect to the components of X must coincide with the derivatives of V, i.e., F must be "pasted smoothly" to V.[11] As an example of using the technique, we review in the appendix A the basic model used by Dixit and Pindyck (1994) in their book (example 1).

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[10] To be exact, we would need a more strict condition on V to make sure that the stopping region is indeed connected. See Dixit and Pindyck (1994), pages 128-130, for the one-dimensional case.

[11] See Øksendahl (2000) for the applicability of the principle in the case where X is defined by a stochastic differential equation. Dixit and Pindyck (1994) provide an intuitive justification (Chapter 4, Appendix C).

2.2 Technological uncertainty

In example 1 (appendix A), the uncertainty is characterized by the geometric Brownian motion, which is a process that continuously fluctuates up and down. Such a process does not, however, seem a proper description of the uncertainty in technological progress. Namely, technological progress is typically driven by innovations that *improve* the current technology. Therefore, using a state variable to model the current level of technological progress, it is natural to think that this variable must be *non-decreasing* in time.[12] With such a process, uncertainty concerns merely the speed at which it grows, not the direction in which it moves. Therefore, in the following we consider how such a refinement changes the problem. By the property that the process $\{X_t^i\}$ is non-decreasing, we mean that the probability $P\left(X_{t_2}^i < X_{t_1}^i\right) = 0$ whenever $t_2 > t_1$.

We now show that this property is very important for the nature of the problem. Namely, assuming that *all* the processes $\{X_t^i\}$ are non-decreasing leads to a dramatic simplification. Remember that when $X \in \Omega$, it is always optimal to invest, i.e., we must have $F(X) = V(X)$ when $X \in \Omega$. Also, as discussed before, Ω is such a region that if $X_1 = \left[X_1^1, ..., X_1^n\right] \in \Omega$, then $X_2 = \left[X_2^1, ..., X_2^n\right] \in \Omega$ whenever $X_2^i \geq X_1^i$ for all $i = 1, ..., n$. This means that if X_t^i are non-decreasing, then X can not exit Ω once it has entered. Then it must also hold at any point of the stopping region including its boundary that:

$$E\left(dF\left(X\right)\right) = E\left(dV\left(X\right)\right) \text{ when } X \in \Omega. \tag{4}$$

From (2), (3), and (4) it then follows that at the boundary it must hold that

$$rV\left(X\right)dt = E\left(dV\left(X\right)\right) \text{ when } X \in \partial\Omega. \tag{5}$$

This is a considerable simplification, because (5) means that it is optimal to invest at such a moment when the opportunity cost of waiting, i.e., the stream of benefit lost within time increment dt due to waiting, $rV\left(X\right)dt$, exceeds the expected change in the value of the investment, $E\left(dV\left(X\right)\right)$. This is similar to the first-order condition of the corresponding deterministic problem, where the expected change in the value of investment is replaced by its actual change. This means that one does not need to account for the probability distribution of the increment in $V\left(X\right)$, it is sufficient to consider the *expected* change of the value. From the solving point of view, the simplification is that one does not need to solve $F\left(X\right)$ in order to find the optimal investment region.[13]

It should be emphasized that this result does *not* mean that the investment decision could be done using the simple net present value method according to which an investment project should be taken whenever its net present value is positive. Such an investment rule is "static" in the sense that it does not account for the development of the project value even if it were deterministic. Thus, our results do not contradict with Farzin et al. (1998), who demonstrate that the optimal pace of technology adoption is optimally slower than implied by the net present value method, because they use that "static" investment rule as the point of comparison, but (5) requires that the current net present value is compared with its *expected improvement*. In our opinion the interpretation of their results can be significantly clarified with (5): it is actually the *expected* rate of technological progress rather

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[12]On the other hand, Grenadier and Weiss (1997) describe technological progress by a state variable that follows the geometric Brownian motion, but their state variable does not directly determine the quality of available technologies. The state variable is used as an indicator that triggers technological improvements.

[13]Assuming that the problem "behaves nicely" so that the first order condition is sufficient, one can determine the boundary of the optimal investment region simply by finding the surface where (5) is satisfied.

than uncertainty that determines whether it is optimal to investment. To confirm this point, we show in the appendix A how their main result is obtained using (5).

An intuitive explanation for the result is that if the stochastic processes are non-decreasing, then there is no option value associated with flexibility, i.e., value in being non-committed to the investment, because there is then no chance that conditions turn more unfavorable in the future. This value of flexibility is a key element in the options methodology, and is the reason why (5) does not normally hold. One can gain further intuitive support by relating the result to a simple two-period setting often given in the literature (e.g. Dixit and Pindyck, 1994, chapter 2). Consider a firm that can trigger an irreversible investment either at period 1 or at period 2, and where there are two possible states of nature at period 2 such that given that the investment was not carried out at period 1, it is optimal to invest if the "good" state is reached, but not if the "bad" state is reached. When considering whether it is optimal to invest at period 1, one must compare the value of the investment at that period with the expected payoff of following the optimal investment policy at period 2. In continuous time, the correspondence of this comparison is the standard Bellman equation (2). However, if the problem is modified so that one knows a priory that it will be optimal to invest in period 2 irrespective of the state of the nature, then it suffices to do the comparison with the expected value of the investment at period 2; one does not need to bother what the optimal investment policy is at period 2. This second case corresponds loosely to our model with non-decreasing processes: if it is optimal to invest at some time instant, it must also be optimal to invest at the "next instant". Thus, there is no need for considering the optimal action at the next instant, and as a result, a condition where the optimal value function F is not present can be used.

Notice that for (4) to hold, it must be that *all* components of X are non-decreasing processes. In one dimension this is trivial, of course. In the appendix A we illustrate the result and its validity with two examples. First, in example 2, we show how the one-dimensional model presented in Farzin et al. (1998) can be solved in a simple way using condition (5). Then, in example 3, we try to apply the result to the model given in example 1, and demonstrate that the technique gives a wrong answer if the stochastic processes are not non-decreasing.

3   Model with technological and revenue related uncertainties

In this section we develop a specific model, where both technological and revenue related uncertainties are present. The purpose is to study how these uncertainties together affect the optimal timing of investment.

The model is motivated by wind power investments. We may envision a given site suitable for wind power production, which does not have any potential alternative use. The site can be understood as an asset, the value of which is contingent on the cost of developing it for wind power production as well as on the revenue stream that such a wind production unit would generate. We consider the problem of choosing the optimal timing to develop the site in order to maximize its value.

We make the following assumptions to characterize the problem. The investment is irreversible, but the timing can be postponed without any constraints. Once the investment has been undertaken, it produces one unit of output per time unit. The only cost that the investor ever faces is the investment cost.[14] However, the

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[14]In other words, we have the simplifying assumption that there are no variable production costs. This is a plausible assumption for wind power production. In reality, however, there are operations and maintenance costs, but we assume that they are fixed and deterministic. Within our model framework fixed flow costs can be included in the investment cost, because the investor

technological progress reduces the investment cost. The level of technological development at time t is, therefore, summarized as the current investment cost, i.e. the amount of money that it would take to build the plant at that time. The technological progress is exogenous, and is driven by innovations that arrive at random times. More specifically, we assume that the investment cost at time $t > 0$ is given by process I_t:

$$I_t = I_0 \phi^{N_t}, \tag{6}$$

where I_0 is the investment cost at time $t = 0$, N_t is a Poisson random variable with mean λt, and $\phi \in [0, 1)$ is a constant reflecting the magnitude of innovations (the smaller the constant, the more each innovation reduces the cost). The parameter λ is the mean arrival rate of innovations. It is easy to confirm that the expected value of I_t is an exponentially declining function of time:

$$E[I_t] = I_0 e^{-\lambda t (1 - \phi)} = I_0 e^{-\gamma t}, \tag{7}$$

where we have denoted $\gamma = \lambda (1 - \phi)$. This means that if the parameters λ and ϕ are adjusted in such a way that the term $\lambda (1 - \phi)$ is kept constant, the expected path of the technological progress is kept unchanged. An increase in the intensity of innovation arrivals (increase in λ) must be compensated by smaller steps (increase in ϕ). However, such adjustments modify the probability distribution of the future investment cost, as can be seen from the variance of I_t:

$$Var[I_t] = I_0^2 \left(e^{-\lambda t \left(1 - \phi^2\right)} - e^{-2\lambda t (1 - \phi)} \right) = I_0^2 \left[\left(e^{-\gamma t} \right)^{(1+\phi)} - \left(e^{-\gamma t} \right)^2 \right]. \tag{8}$$

It is easy to see from (8) that if λ is increased towards ∞, and ϕ is correspondingly increased towards 1 in such a manner that the term $\gamma = \lambda (1 - \phi)$ is kept constant, then $Var[I_t]$ approaches zero. Thus, uncertainty can be decreased by increasing λ and ϕ, the limiting case being deterministic exponential decline: $I_t = I_0 e^{-\gamma t}$. On the other hand, the highest level of uncertainty is obtained when ϕ is reduced to zero, which corresponds to the case where the investment cost collapses to zero at the first innovation. These opposing two cases will be considered later as examples.

We denote the price of output at time t by P_t. In case of wind power production, it includes the electricity price plus the possible additional revenue that the plant owner receives, for example the price of tradeable green certificates. We assume that once the plant has been built, it produces the revenue stream P_t forever.[15] P_t is a stochastic process that we assume to follow the geometric Brownian motion:

$$\frac{dP_t}{P_t} = \mu dt + \sigma dz, \tag{9}$$

where μ and σ are constants reflecting the drift and volatility of the process, and dz is the standard Brownian motion increment. We assume that $0 \le \mu < r$, where r is the risk free rate of return. From the properties of the geometric Brownian motion, it follows directly that the expected value of the price at some future time is:

$$E[P_t] = P_0 e^{\mu t}. \tag{10}$$

The degree of revenue uncertainty can be adjusted by changing σ. As seen from (10), such changes do not change the expected price at a given future time.

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can invest a sufficient sum of money in bonds, for example, in order to ensure a sufficient flow of income to cover the subsequent deterministic flow costs.

[15] Infinete life time is assumed for simplicity. In reality, the lifetime of a wind turbine is typically around 20 years.

When the plant has been built, its value at time t is the expected discounted sum of cash flows it produces in the future:[16]

$$g\left(P_t\right) = E\left[\int\limits_{s=t}^{\infty} P_s e^{-r(s-t)} ds\right] = \int\limits_{s=t}^{\infty} P_t e^{\mu(s-t)} e^{-r(s-t)} ds = \frac{P_t}{r-\mu}. \qquad (11)$$

At the time of investment, the investor pays I_t to obtain the plant. The payoff of the investment is thus $V\left(P_t, I_t\right) = \frac{P_t}{r-\mu} - I_t$. Obviously, the problem is of the same form as discussed in section 2. Denoting $X_t^1 = P_t$ and $X_t^2 = \left(I_t\right)^{-1}$, for example, it is easy to confirm that all assumptions are satisfied by $\{X_t\}$ and V. The calendar time does not affect the problem, so we can leave the subscripts out and denote the state of the model simply as (P, I). The value of the investment is thus $V\left(P, I\right) = \frac{P}{r-\mu} - I$, and the value of the option to invest is:

$$F\left(P, I\right) = \sup_\tau E\left[e^{-r\tau}\left(\frac{P_\tau}{r-\mu} - I_\tau\right)\right], \qquad (12)$$

where P_τ and I_τ refer to the output price and investment cost at some future time τ when the processes start from P and I and evolve according to (6) and (9).

As discussed in section 2, the solution to the problem must be a stopping region in the (P, I)-space, which we denote by Ω. Notice that one can enter Ω in two ways: either by continuous diffusion of P or by a sudden jump of I. We propose next that Ω has a particularly simple form.

Proposition 1 *The optimal stopping region Ω must be of the form:*

$$\Omega = \left\{(P, I)\left|\frac{P}{I} \geq p^*\right.\right\},$$

where p^* is a constant to be determined.

Proof. In the appendix B. ∎

The key to the result is the observation from equation (12) that $F\left(kP, kI\right) = kF\left(P, I\right)$, i.e. F is homogenous of degree one in (P, I). Thus, dividing F by I results $\frac{F(P,I)}{I} = F\left(\frac{P}{I}, 1\right)$, which is a function of the fraction of P and I only. The model is simplified considerably by denoting this fraction by a new variable. We adopt the following definitions, similar to Dixit and Pindyck (1994), section 6.5:

$$p \equiv \frac{P}{I}, \qquad (13)$$

$$f\left(p\right) \equiv F\left(p, 1\right). \qquad (14)$$

Expressed in another way, (14) means that $F\left(P, I\right) = If\left(p\right)$. The new variable p follows the combined geometric Brownian motion - jump process:

$$\frac{dp}{p} = \mu dt + \sigma dz + dq, \qquad (15)$$

where

$$dq = \left\{\begin{array}{ll} 0 & \text{with probability } 1 - \lambda dt, \\ \left(\phi^{-1} - 1\right) & \text{with probability } \lambda dt. \end{array}\right. \qquad (16)$$

---

[16]We assume risk-neutrality here. However, we may alternatively assume that the fluctuations in P are spanned by traded assets. In that case we interpret the problem using equivalent risk-neutral valuation. Then equation (9) is given under the martingale measure, in which case the growth rate μ differs from the "real" growth rate.

We know from proposition 1 that the optimal solution is to invest at the first moment when p rises above some threshold level p^*. The problem is to find this threshold. With this in mind, we return to the original problem (12). The Bellman equation (2) is:

$$rF(P, I) dt = E(dF(P, I)), \text{ when } (P, I) \notin \Omega, \tag{17}$$

where $dF(P, I)$ is the infinitesimal change in $F(P, I)$ given P and I evolve according to equations (6) and (9). Using Ito's lemma and the fact that I drops to ϕI at probability λdt within infinitesimal time increment dt, we obtain the following expression for $E(dF(P, I))$:

$$E(dF(P, I)) = (r - \delta) P F_P dt + \frac{1}{2} \sigma^2 P^2 F_{PP} dt + \lambda [F(P, \phi I) - F(P, I)] dt, \tag{18}$$

where F_P and F_{PP} refer to the first and second derivatives of F with respect to P evaluated at (P, I). Dividing by dt and arranging terms, (17) becomes the following partial differential equation:

$$\frac{1}{2} \sigma^2 P^2 F_{PP} + (r - \delta) P F_P - rF(P, I) + \lambda [F(P, \phi I) - F(P, I)] = 0, \tag{19}$$

Using (13) and (14), we obtain:

$$F_P = I \frac{\partial f(p)}{\partial p} \frac{\partial p}{\partial P} = I f'(p) \frac{1}{I} = f'(p), \tag{20}$$

$$F_{PP} = \frac{\partial f'(p)}{\partial p} \frac{\partial p}{\partial P} = \frac{f''(p)}{I}, \tag{21}$$

$$F(P, \phi I) = \phi I f\left(\frac{P}{\phi I}\right) = \phi I f\left(\frac{p}{\phi}\right), \tag{22}$$

where $f'(p)$ and $f''(p)$ denote the first and second derivatives of f evaluated at p. Substituting (13), (14), and (20) - (22) in (19) and dividing by I results in:

$$\frac{1}{2} \sigma^2 p^2 f''(p) + \mu p f'(p) - (r + \lambda) f(p) + \lambda \phi f\left(\frac{p}{\phi}\right) = 0. \tag{23}$$

The value of the investment option is thus a function $F(P, I) = I f(p)$ such that $f(p)$ satisfies (23) whenever $(P, I) \notin \Omega$, i.e. whenever $p < p^*$. In addition, the value function F must satisfy the following boundary conditions at the boundary $p = \frac{P}{I} = p^*$:

$$F(P, I) = \frac{P}{r - \mu} - I, \text{ when } \frac{P}{I} = p^*, \tag{24}$$

$$F_P = \frac{1}{r - \mu}, \text{ when } \frac{P}{I} = p^*. \tag{25}$$

The first one is the value matching condition and the second one is the smooth pasting condition. The conditions can be written in terms of f and p:[17]

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[17] In fact, there is also a second smooth-pasting condition: $F_I = -1$. Since $F_I = f(p) + I f'(p) \frac{-P}{I^2} = f(p) - p f'(p)$, this condition can be written as $f(p^*) = p^* f'(p^*) - 1$. It is easy to see that the condition is already implied by (26) and (27).

$$f(p^*) = \frac{p^*}{r-\mu} - 1, \qquad (26)$$

$$f'(p^*) = \frac{1}{r-\mu}. \qquad (27)$$

Moreover, if p approaches zero, the value of the investment opportunity must approach zero. This adds one more condition:

$$\lim_{p \to 0^+} f(p) = 0. \qquad (28)$$

There is a special difficulty characteristic to our problem that can be detected by looking more carefully at (23). Namely, the differential equation (23) is not "local" to the point p, because it contains the term $f\left(\frac{p}{\phi}\right)$, which is the value of the solution function at the point $\frac{p}{\phi} > p$. When p is sufficiently close to p^*, we have $\frac{p}{\phi} > p^*$, and (23) contains the value of $f(\cdot)$ *inside* the stopping region. The reason for this property of the problem is that the innovations may move the state directly across the boundary between stopping and continuation regions, which leads to an immediate investment. Whenever $p > p^*$, (3) gives $F(P, I) = \frac{P}{\delta} - I$. This means that:

$$f(p) = \frac{p}{r-\mu} - 1 \;\; \forall p > p^*. \qquad (29)$$

Having this, the problem is well defined. One must find a real valued function f and a positive number p^* such that f satisfies (23) when $p \leq p^*$, (29) when $p > p^*$, (26) and (27) at $p = p^*$, and (28) when $p \to 0$. Unfortunately, even if it can be shown that a unique solution for p^* exists, there is no closed form solution for $f(p)$ that would satisfy all the conditions. It is possible to construct a numerical procedure for solving p^*,[18] but instead of going into that we concentrate in the next section on several special cases, which can be solved analytically. Together, they capture the main insights of the paper.

4 Analytic solutions

In order to see how the two uncertainties affect the problem, we consider several special cases that are obtained by adjusting the parameters σ, λ, and ϕ in such a way that the expected price and investment cost paths remain unchanged. The value of σ does not change the expected value of future output price as seen from (10), so σ can be varied directly. The greater the volatility σ, the greater the revenue uncertainty. However, to adjust λ and ϕ in a way that does not affect the expected future investment cost, we must keep $\gamma = \lambda(1-\phi)$ fixed. The greater the parameter ϕ, the greater the parameter λ, and the smaller the technological uncertainty.

Thus, in all three cases that follow, we have $E[P_t] = P_0 e^{\mu t}$ and $E[I_t] = I_0 e^{-\gamma t}$, where γ and μ are fixed constants. Since the value of the investment, $V(P, I) = \frac{P}{r-\mu} - I$, is a linear combination of P and I, its expected value also remains unchanged, namely:

$$E[V(P_t, I_t)] = \frac{P_0}{r-\mu} e^{\mu t} - I_0 e^{-\gamma t}. \qquad (30)$$

---

[18] A sketch of such a procedure available from the author.

4.1 Special case A: deterministic price process

First, we consider the special case where $\sigma = 0$ in equation (9). Then P_t is a deterministic increasing function of time given by

$$P_t = P_0 e^{\mu t}, \tag{31}$$

where P_0 is the initial value of P. Denoting $X^1 = P$, and $X^2 = I^{-1}$, the problem is of the form discussed in section 2. Moreover, since P is now deterministic and increasing, both of the processes $\{X_t^1\}$ and $\{X_t^2\}$ are non-decreasing. Therefore, we can use the condition (5) to find the boundary of Ω. In section 3 we stated that the boundary must be of the form $\partial\Omega = \left\{(P,I) \,\middle|\, \frac{P}{I} = p^*\right\}$, where p^* is a constant to be determined.

The expected change in V given the state (P, I) is:

$$E\left(dV\left(P,I\right)\right) = \frac{\mu P}{r - \mu} dt + \lambda dt \left(1 - \phi\right) I = \left[\frac{\mu P}{r - \mu} + \gamma I\right] dt. \tag{32}$$

Therefore, condition (5) says that:

$$r\left(\frac{P}{r - \mu} - I\right) dt = \left[\frac{\mu P}{r - \mu} + \gamma I\right] dt. \tag{33}$$

This simplifies to $P/I = r + \gamma$. Thus, denoting by p^A the optimal threshold (A for special case A), the solution is:

$$p^A = r + \gamma. \tag{34}$$

Note that p^A depends on λ and ϕ only through γ, which is assumed fixed. This means that the degree of uncertainty does not affect the decision of whether to invest or not given the current price and investment cost. The decision maker can replace the actual investment cost process with its expected value. In more practical terms, an estimate of the time trajectory of the future investment cost is sufficient for the correct investment decision as long as it is unbiased (i.e. it represents the expected value of the process). There is no need for an estimate concerning the level of uncertainty associated with the process.

To gain more insight, we can also derive the result directly using (23). Since $\sigma = 0$, (23) becomes:

$$\mu p f'\left(p\right) - \left(r + \lambda\right) f\left(p\right) + \lambda\phi f\left(\frac{p}{\phi}\right) = 0. \tag{35}$$

At the threshold p^A, (26) and (27) must hold:

$$f\left(p^A\right) = \frac{p^A}{\left(r - \mu\right)} - 1, \tag{36}$$

$$f'\left(p^A\right) = \frac{1}{\left(r - \mu\right)}. \tag{37}$$

Further, from (29) we have

$$f\left(\frac{p^A}{\phi}\right) = \frac{p^A}{\phi\left(r - \mu\right)} - 1. \tag{38}$$

At the optimal threshold (35) must be satisfied together with (36) - (38). Substituting (36) - (38) in (35) yields directly the equation (34).

Note that we did not need the condition (28) to get the result. The reason for this can be explained as follows. Obviously, sufficiently close to the point p^A the

next innovation would move p directly across the investment threshold triggering the investment. When the output price process is increasing and deterministic, then once p climbs sufficiently close to p^A for this to be the case, then it is sure that this will be the case ever after. Thus, only the upper boundary conditions at p^A are relevant, and the problem is uniquely solved using them. On the contrary, if the price process would be stochastic, price could always go down moving p away from the threshold level, and thus the solution depends also on what happens at lower values of p, which makes condition (28) relevant.

4.2 Special case B: deterministic technological progress

In this case, the intensity of innovation arrivals is increased ($\lambda \to \infty$) and the step size is reduced ($\phi \to 1$) so that the investment cost process approaches deterministic exponential decline:

$$I_t = e^{-\gamma t}, \tag{39}$$

where the limiting processes $\lambda \to \infty$ and $\phi \to 1$ are such that $\gamma = \lambda(1 - \phi)$ is fixed.

The solution to the problem can be derived directly from (23), which we rewrite for convenience:[19]

$$\frac{1}{2}\sigma^2 p^2 f''(p) + \mu p f'(p) - (r + \lambda) f(p) + \lambda \phi f\left(\frac{p}{\phi}\right) = 0. \tag{40}$$

We consider what happens to the term $\lambda \phi f\left(\frac{p}{\phi}\right)$ when $\lambda \to \infty$ and $\phi \to 1$. Expanding the term about the point p results that close to p, i.e. with ϕ close to 1 we have:

$$
\begin{aligned}
\lambda \phi f\left(\frac{p}{\phi}\right) &= \lambda \phi \left[f(p) + \left(\frac{p}{\phi} - p\right) f'(p) + \frac{1}{2}\left(\frac{p}{\phi} - p\right)^2 f''(p) + ... \right] \\
&= \lambda \phi f(p) + p \gamma f'(p) + \frac{1}{2}p^2 \gamma \left(\frac{1}{\phi} - 1\right) f''(p) + ..., \tag{41}
\end{aligned}
$$

where we have omitted terms of higher order than 2. As $\phi \to 1$, the term $\frac{1}{2}p^2 \gamma \left(\frac{1}{\phi} - 1\right) f''(p)$ vanishes (as do all the higher order terms, which can be easily shown), and thus we find that $\lambda \phi f\left(\frac{p}{\phi}\right) \to \lambda \phi f(p) + p \gamma f'(p)$ when $\phi \to 1$, $\lambda = \frac{\gamma}{(1-\phi)}$. Substituting this in (40) and replacing $\lambda(1 - \phi)$ by γ results in:

$$\frac{1}{2}\sigma^2 p^2 f''(p) + (\mu + \gamma) p f'(p) - (r + \gamma) f(p) = 0. \tag{42}$$

This is a standard second-order differential equation with the general solution:

$$f(p) = B_1 p^{\beta_1^B} + B_2 p^{\beta_2^B}, \tag{43}$$

where B_1 and B_2 are constant parameters and

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[19] Alternatively, the solution could be derived by directly using the deterministic process $I_t = I_0 e^{\gamma t}$ for the investment cost. Then the only source of uncertainty is the output price that follows the geometric Brownian motion, and thus the techniques reviewed in Dixit and Pindyck (1994) could be applied. In fact, the problem could be transformed to the same form as example 1 by noting that the solution must be the same as if the investment cost is assumed constant, discount factor is increased by γ, and the growth rate of output price, μ, is increased by γ. We derive the solution by taking the limiting processes for the equation (23) in order to emphasize that we have a special case of the problem presented in section 3.

$$\beta_1^B = \frac{1}{2} - \frac{(\mu + \gamma)}{\sigma^2} + \sqrt{\left[\frac{(\mu + \gamma)}{\sigma^2} - \frac{1}{2}\right]^2 + \frac{2(r + \gamma)}{\sigma^2}} > 1, \tag{44}$$

$$\beta_2^B = \frac{1}{2} - \frac{(\mu + \gamma)}{\sigma^2} - \sqrt{\left[\frac{(\mu + \gamma)}{\sigma^2} - \frac{1}{2}\right]^2 + \frac{2(r + \gamma)}{\sigma^2}} < 0. \tag{45}$$

Condition (28) implies that $B_2 = 0$. Conditions (26) and (27) are for (43):

$$B_1 \left(p^B\right)^{\beta_1^B} = \frac{p^B}{(r - \mu)} - 1, \tag{46}$$

$$B_1 \beta_1^B \left(p^B\right)^{\beta_1^B - 1} = \frac{1}{(r - \mu)}. \tag{47}$$

These are easily solved for the two unknowns, B_1 and p^B:

$$B_1 = \left[\beta_1^B (r - \mu)\right]^{-\beta_1^B} \left(\beta_1^B - 1\right)^{\left(\beta_1^B - 1\right)}, \tag{48}$$

$$p^B = \frac{\beta_1^B}{\beta_1^B - 1} (r - \mu), \tag{49}$$

where β_1^B is given by (44).

4.3 Special case C: full collapse of the investment cost ($\phi = 0$)

In this case we assume that $\phi = 0$. To still keep $\gamma = \lambda(1 - \phi)$ fixed, we must have $\lambda = \gamma$. This means that the probability of an innovation in the near future is low, γdt within an infinitesimal time increment, but once an innovation occurs, the investment becomes completely free. After such a break-through innovation it is optimal to invest immediately, because the project does not cost anything, but will provide a positive income stream forever. This means that the value of the option to invest at cost zero is the same as the value of the project, that is, $F(P, 0) = \frac{P}{r - \mu}$. The expected change in the value of the investment, as given by equation (18) is thus:

$$E(dF) = \mu P F_P dt + \frac{1}{2} \sigma^2 P^2 F_{PP} dt + \lambda \left[\frac{P}{r - \mu} - F(P, I)\right] dt. \tag{50}$$

Following the same steps as in section 3, i.e. substituting (50) in (17), dividing by dt, arranging terms, using (13), (14), and (20) - (22), and finally dividing by I results in:

$$\frac{1}{2} \sigma^2 p^2 f''(p) + \mu p f'(p) - (r + \gamma) f(p) + \gamma \frac{p}{r - \mu} = 0. \tag{51}$$

The general solution to (51) is:

$$f(p) = C_1 p^{\beta_1^C} + C_2 p^{\beta_2^C} + \frac{\gamma p}{(r - \mu)(r - \mu + \gamma)}, \tag{52}$$

where C_1 and C_2 are constant parameters and

$$\beta_1^C = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left[\frac{\mu}{\sigma^2} - \frac{1}{2}\right]^2 + \frac{2(r + \gamma)}{\sigma^2}} > 1, \tag{53}$$

$$\beta_2^C = \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{\left[\frac{\mu}{\sigma^2} - \frac{1}{2}\right]^2 + \frac{2(r + \gamma)}{\sigma^2}} < 0. \tag{54}$$

Again, (28) implies that $C_2 = 0$. The conditions (26) and (27) are for (52):

$$C_1 \left(p^C\right)^{\beta_1^C} + \frac{\gamma p^C}{(r - \mu)(r - \mu + \gamma)} = \frac{p^C}{r - \mu} - 1, \tag{55}$$

$$C_1 \beta_1^C \left(p^C\right)^{\beta_1^C - 1} + \frac{\gamma}{(r - \mu)(r - \mu + \gamma)} = \frac{1}{r - \mu}. \tag{56}$$

These can be solved for the two unknowns, C_1 and p^C:

$$C_1 = \left[\beta_1^C (r - \mu + \gamma)\right]^{-\beta_1^C} \left(\beta_1^C - 1\right)^{\left(\beta_1^C - 1\right)}, \tag{57}$$

$$p^C = \frac{\beta_1^C}{\beta_1^C - 1} (r - \mu + \gamma), \tag{58}$$

where β_1^C is given by (53).

4.4 Summing up

Special case A would suggest that the technological uncertainty simply does not affect the optimal investment rule. However, if this was generally true, then the investment thresholds p^B and p^C given by equations (49) and (58) should be equal. Figure 1 shows these thresholds as functions of σ with parameter values $r = 0.05$, $\mu = 0.01$, and $\gamma = 0.05$. It can be seen that as the special case A indicates, these coincide when $\sigma = 0$. However, as σ is increased, we have $p^C > p^B$. In other words, when there is no revenue uncertainty, then the technological uncertainty does not affect the optimal investment threshold, but when revenue uncertainty is added in the model, the technological uncertainty also starts to affect making p^B and p^C depart from each other.

This result can be explained by the discussion of section 2. When output price is deterministic, we have the case where both state variables are non-decreasing (P is deterministic and increasing, and I^{-1} is a non-decreasing stochastic process). Therefore, the condition (5) must be satisfied at the boundary of the stopping region, which implies that uncertainty does not affect the optimal investment threshold. However, as revenue uncertainty is added in the model, then P is no longer non-decreasing. Then (4) does not hold any longer, and thus we can not use (5) to determine the stopping region. Then it is not surprising that uncertainty in P affects the optimal stopping region, as confirmed by the fact that curves p^B and p^C are increasing. It is perhaps more surprising that even uncertainty in I can not be ignored any longer when P is stochastic, as confirmed by the fact that $p^B \neq p^C$.

As mentioned in section 3, the general model can not be solved analytically. However, to complete the characterization of the solution, we state a proposition, which implies that for intermediate values of uncertainty in the investment cost (i.e., with $\phi \in (0, 1)$), the threshold level for p is between the two special cases:

Proposition 2 *Keeping $\gamma = \lambda(1 - \phi)$ fixed, the optimal investment threshold p^* is decreasing in λ and ϕ, and thus increasing in the degree of technological uncertainty.*
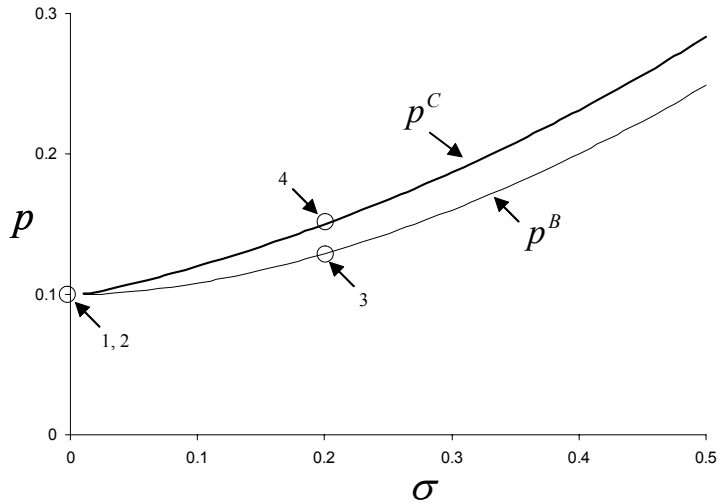
Figure 1: Investment thresholds in special cases B and C as functions of σ.

Proof. In the appendix B. ■

Since at the lowest and highest possible values of ϕ ($\phi = 0$ and $\phi \to 1$) we have $p^* = p^B$ and $p^* = p^C$, respectively, it is clear that generally the optimal investment threshold p^* with $\phi \in (0, 1)$ is between p^B and p^C. Thus, the region between the curves p^B and p^C in figure 1 represents the area of possible investment thresholds at all such combinations of σ, λ, and ϕ that keep the expected paths of P and I fixed.

For further illustration, we show how the value function f is affected by technological- and revenue uncertainties. Figure 2 shows this function at four different combinations of uncertainty related parameters σ, λ, and ϕ (other parameters are as in figure 1: $r = 0.05$, $\mu = 0.01$, and $\gamma = 0.05$):

1. $f_1 : \sigma = 0$, $\lambda \to 0$, and $\phi \to 1$,

2. $f_2 : \sigma = 0$, $\lambda = 0.05$, and $\phi = 0$,

3. $f_3 : \sigma = 0.2$, $\lambda \to 0$, and $\phi \to 1$,

4. $f_4 : \sigma = 0.2$, $\lambda = 0.05$, and $\phi = 0$.

The curves f_1 and f_3 are calculated from equation (43) and the curves f_2 and f_4 from equation (52). We denote the corresponding investment thresholds by $p_1, ..., p_4$. These four cases are also marked in figure 1 with numbers $1, ..., 4$.

The curve f_1 represents the value function when there is neither technological- nor revenue uncertainty. The curve f_2 shows what happens when revenue uncertainty is maintained at zero level, but technological uncertainty is increased to the maximum level. It can be seen that the value of the investment opportunity is increased in the region where it is optimal to wait. In other words, the option value of investment is increased by the increase of uncertainty, which is a standard result in the real options theory. However, the shapes of the curves are such that they move closer to each other when approaching the stopping region. Finally, they reach each other at a common investment threshold. Both of these cases belong to the special case A, so the investment threshold in both cases is given by (34), and has the value $p_1 = p_2 = 0.1$.
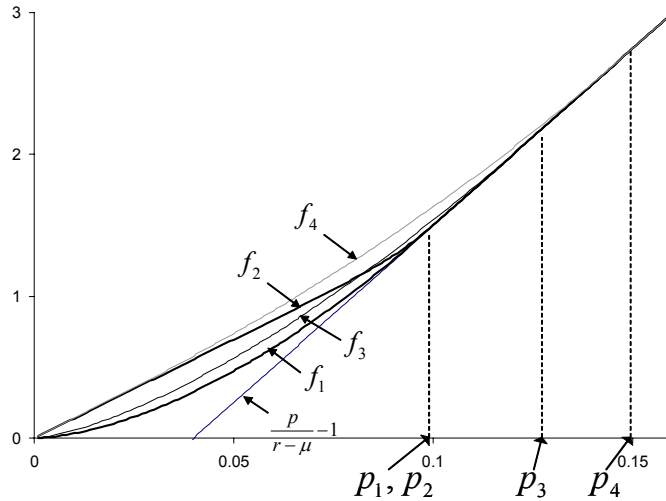
Figure 2: Value functions at different combinations of σ, λ, and ϕ.

On the contrary, the curve f_3 represents the case where technological uncertainty is absent, but instead, the revenue uncertainty is increased. Again, uncertainty increases the option value of waiting. However, the shape of f_3 differs from that of f_2. In contrast to the technological uncertainty, the extra value created by revenue uncertainty increases when moving towards the stopping region. This implies that the threshold p_3 departs from p_1 and p_2. The exact value in this case is $p_3 = 0.129$.

Finally, the curve f_4 represents the case where both uncertainties are present. As with f_3, the revenue uncertainty has the effect that p_4 is higher than p_1 and p_2. However, now that it is combined with revenue uncertainty, the technological uncertainty has an additional effect, which moves p_4 even further up than p_3, the exact value being $p_4 = 0.15$.

To sum up, the comparison of f_1 and f_2 reveals that in the absence of revenue uncertainty, the degree of technological uncertainty increases the value of the investment option, but does not affect the optimal investment rule. However, when combined with revenue uncertainty, also the optimal investment rule is affected, as can be seen by comparing f_3 and f_4.

5   Conclusions

We have studied the timing of investment under uncertain technological progress and uncertain revenue stream. The analysis was based on the theory of irreversible investment under uncertainty. Most of the existing literature considers uncertainty in input or output prices leaving technological uncertainty with little attention. Our methodological contribution is to study the interaction of both of these uncertainties.

We first characterized the general problem of investment in the optimal stopping framework, and showed that if the underlying stochastic processes are non-decreasing, an intuitive optimality condition similar to the first-order condition of the corresponding deterministic problem can be used. This means that the optimal investment rule depends only on the expected growth of the net present value of investment. We then presented a specific model where two uncertainties are present. First, there is technological uncertainty, where innovations arrive at exponentially

distributed random times reducing the cost of investment. Second, the revenue stream that the investment would generate fluctuates according to the geometric Brownian motion, making the present value of the investment a stochastic process. When only the technological uncertainty is present, then an estimate on the expected path of future development is sufficient for the optimal investment timing decision, as explained by the preceding analysis. However, we found that when revenue uncertainty is included, then also the technological uncertainty starts to affect the investment decision making the investor more hesitant to undertake the project. Thus, it is the combination with the revenue uncertainty that makes the technological uncertainty relevant for the decision maker.

For analytical convenience, we have used rather coarse stochastic processes to characterize the output price movements and technological progress. For example, the sizes of the investment cost reductions due to innovations would more realistically be random variables. However, the processes we used are sufficient for our purpose, which is to characterize the effects of the two types of uncertainties. They capture the main properties of revenue uncertainty and technological progress, namely the revenue that moves randomly in both directions and technological uncertainty that concerns the speed at which technology improves. Even with a more refined model for these uncertainties, the qualitative nature of the results is likely to remain the same. However, in a real decision making application, for example in connection with a development of a given wind farm, a thorough identification of the processes and estimation of the parameters would be necessary. The solving would require more tailored numerical methods.

A Appendix

Example 1 *In this example, we review the basic model used by Dixit and Pindyck (1994) in their book. A more refined version of the model was originally presented in McDonald and Siegel (1986).*

Assume that $\{X_t\}$ is a one-dimensional process that follows the geometric Brownian motion:

$$dX = \alpha X dt + \sigma X dz, \tag{59}$$

where α and σ are positive constants and dz is the standard Brownian motion increment. X represents the present value of the investment, and the cost of investment is constant I. Thus, the net present value of the investment is $V(X) = X - I$.[20] In order to have a solution to the problem, it must be that α is lower than the discount factor r (otherwise the value of the project grows so fast that it is always optimal to wait).

Using Ito's lemma, the expected value of the change in $F(X)$ given (59) is:

$$E(dF(X)) = \alpha X F'(X) dt + \frac{1}{2}\sigma^2 F''(X) dt,$$

where primes denote derivatives with respect to X. Substituting this in (2) and dividing by dt yields the following differential equation:

$$\frac{1}{2}\sigma^2 F''(X) + \alpha X F'(X) - rF(X) = 0. \tag{60}$$

Since we are in one dimension, the stopping region must be of the form $\Omega = (X^*, \infty)$, and the problem is to find the optimal investment threshold $\partial\Omega = X^*$. The condition (3) at the boundary X^* is thus:

$$F(X^*) = X^* - I. \tag{61}$$

The smooth-pasting condition is:

$$F'(X^*) = 1. \tag{62}$$

In addition, since 0 is a absorbing barrier for X, we must have the condition that in the limit where $X \to 0$, the value of the investment option goes to zero as well. This means that the solution to (60) must be of the form:

$$F(X) = AX^\beta, \tag{63}$$

where A is a constant to be determined and

$$\beta = \frac{1}{2} - \frac{\alpha}{\sigma^2} + \sqrt{\left(\frac{\alpha}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} > 1. \tag{64}$$

Using the conditions (61) and (62), the parameter A and the optimal investment threshold can be easily solved. We get for the latter:

$$X^* = \frac{\beta}{\beta - 1}I. \tag{65}$$

Since $\beta > 1$, X^* exceeds I by a certain cap. This means that the investor should wait until the present value of the project exceeds the cost of investment by a strictly positive amount. It is easy to show that X^* is increasing in σ. In other words, the higher the uncertainty, the higher the net present value of the project must be to make investing optimal.

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[20]Our notation differs from Dixit and Pindyck (1994). They use V for the present value of the investment and assume that it follows the geometric Brownian motion. The net present value in their model is thus $V - I$, corresponding to our $X - I$.

Example 2 *The model of Farzin et al. (1998) considers the optimal timing of tech-nology adoption. A firm uses an old production technology, but has an opportunity to adopt a newer technology at a given cost. The technology evolves in time, and the problem of the firm is to choose the optimal time to switch. In the basic version of the model, only one switch is allowed. The technological progress is described by the parameter θ, which is subject to jumps due to technological innovations that arrive randomly. More precisely, θ follows a Poisson jump process so that within an infinitesimal time increment dt, the change in θ is*

$$d\theta = \begin{cases} u & \text{with probability } \lambda dt, \\ 0 & \text{with probability } 1 - \lambda dt, \end{cases} \tag{66}$$

where u is a random variable uniformly distributed over the interval $(0, \overline{u})$. Thus, both the time and the extent of the next innovation are random.

The firm produces a homogeneous good according to the production function

$$h(v, \theta) = \theta v^a, \tag{67}$$

where v is a variable input and a is the constant output elasticity. The unit cost of the variable input, w, and the output price, p, are assumed constant. Farzin et al. derive the present value of the cash flows of a firm that produces using technology θ forever:

$$g(\theta) = \frac{\varphi \theta^b}{r}, \tag{68}$$

where $\varphi = (1 - a)(a/w)^{a/(1-a)} p^{1/(1-a)}$ and $b = 1/(1 - a) > 1$ are constants and r is the discount factor.

The firm is initially using technology θ_0 and has an opportunity to make a single switch to a newer technology at a cost I. If the level of technology is θ when the firm switches, the value of the switch is thus $\frac{\varphi \theta^b}{r} - \frac{\varphi \theta_0^b}{r}$. To put the problem in our investment framework, the firm has thus an option to carry out an irreversible investment whose net present value V depends on the stochastic variable θ as given below:

$$V(\theta) = \frac{\varphi \theta^b}{r} - \frac{\varphi \theta_0^b}{r} - I. \tag{69}$$

Obviously, V is an increasing function of θ, and θ is a time-homogenous Markov process. Thus, we know that the optimal solution to the problem must be to invest at the first such moment when θ exceeds some threshold value θ^*. Moreover, since θ is a non-decreasing process, we can apply the condition (5) to find the optimal θ^*. In the present case, (5) can be written:

$$rV(\theta^*) dt = E(dV(\theta^*)). \tag{70}$$

Since an innovation occurs at probability λdt within an infinitesimal dt, the expected change in V given the current θ is:

$$E(dV(\theta)) = \lambda dt \left[\int_0^{\overline{u}} \left(\frac{\varphi(\theta + u)^b}{r} - \frac{\varphi \theta_0^b}{r} - I \right) \frac{1}{\overline{u}} du - \left(\frac{\varphi \theta^b}{r} - \frac{\varphi \theta_0^b}{r} - I \right) \right]$$

$$= \lambda dt \left[\frac{\varphi}{\overline{u}r} \left(\frac{(\theta + \overline{u})^{1+b} - \theta^{1+b}}{1 + b} \right) - \frac{\varphi \theta^b}{r} \right]. \tag{71}$$

Substituting this in (70) and simplifying yields:

$$\frac{\lambda \varphi}{\overline{u}r} \left(\frac{(\theta^* + \overline{u})^{1+b} - (\theta^*)^{1+b}}{1 + b} \right) - \frac{(r + \lambda)\varphi}{r} (\theta^*)^b + \varphi \theta_0^b + rI = 0. \tag{72}$$

This is exactly the same as equation (17) in Farzin et al. We have thus shown that their result is obtained in a simple and intuitive way by using condition (5). The optimal investment threshold is obtained by solving (72) numerically.

Example 3 (Example 1 continued) *We try to apply the result (5) in example 1, i.e. invest at the moment when:*

$$rV\left(X^{*}\right)dt = E\left(dV\left(X^{*}\right)\right). \tag{73}$$

Since $rV\left(X\right)dt = r\left(X - I\right)dt$ and $E\left(dV\left(X\right)\right)) = \alpha X dt$, we find that (73) implies:

$$X^{*} = \frac{r}{r - \alpha}I. \tag{74}$$

It can be shown that $\frac{\beta}{\beta-1} > \frac{r}{r-\alpha}$ when $r > \alpha$, thus (74) is clearly wrong. In fact, (74) would be the correct investment threshold if $\sigma = 0$, i.e., the process for X would be deterministic. The reason why the result (5) does not work for the present case is that the geometric Brownian motion fluctuates both up and down. Thus, (4) does not hold at the boundary of the stopping region, and therefore (73) can not be used to find the correct investment threshold.

B Appendix

Proof of proposition 1. We have to show that the optimal stopping region Ω of the problem (12) must be of the form $\Omega = \left\{ (P, I) \left| \frac{P}{I} \geq p^* \right. \right\}$ where p^* is a constant. The proof follows a similar proof in McDonald and Siegel (1986).

Let τ^* be the stopping time that maximizes the expression

$$E\left[\left(\frac{P}{r - \mu} - I \right) e^{-r\tau} \right], \tag{75}$$

where P and I follow (9) and (6) respectively. Let Ω be the corresponding stopping region. Write $P' = kP$, $I' = kI$ ($k > 0$), and consider problem to choose τ' to maximize

$$E\left[\left(\frac{P'}{r - \mu} - I' \right) e^{-r\tau} \right] = kE\left[\left(\frac{P}{r - \mu} - I \right) e^{-r\tau} \right] \tag{76}$$

subject to the appropriate stochastic processes for P' and I'. It is easy to confirm that the processes are exactly the same as those for P and I (properties of the geometric Brownian motion and Poisson jump processes). Therefore, the problem is the same as that in (75), so the solution stopping region is again Ω. Because the expression in (76) is just the expression (75) multiplied by k, the optimal stopping times must be equal, i.e. $\tau' = \tau^*$. So, in every realization of P and I, the pairs (P, I) and (kP, kI) hit Ω at the same time. Since k is an arbitrary positive constant, it must be that the boundary of Ω is a ray originating from origin (it will be shown in the next step that there are not many such rays). We denote the ratio of $\frac{P}{I}$ at this stopping boundary by p^*.

Assume that the boundary Ω consists of many rays. Then there must be a $p' = \frac{P'}{I'} > p^* = \frac{P^*}{I^*}$ where it is not optimal to invest. Then it must be that $\frac{P^*}{r-\mu} - I^* = F(P^*, I^*)$, but $\frac{P'}{r-\mu} - I' < F(P', I')$. These can be written as $\frac{p^*}{r-\mu} - 1 = F(p^*, 1)$ and $\frac{p'}{r-\mu} - 1 < F(p', 1)$. Multiplying the second by $\frac{p^*}{p'}$ gets $\frac{p^*}{r-\mu} - \frac{p^*}{p'} < F\left(p^*, \frac{p^*}{p'}\right)$, which means that it is not optimal to pay $\frac{p^*}{p'}$ to get the flow $\frac{p^*}{r-\mu}$. However, according to equation $\frac{p^*}{r-\mu} - 1 = F(p^*, 1)$, it is optimal to pay 1 to get the flow $\frac{p^*}{r-\mu}$. This is contradicting, because $\frac{p^*}{p'} < 1$. Thus, the unique solution to the problem must be to invest whenever $p \in [p^*, \infty)$, and to wait otherwise. ∎

Proof of proposition 2. We have to show that p^* is decreasing in ϕ (when keeping γ constant). This would imply that the optimal investment threshold p^* with $\phi \in (0, 1)$ is between p^B and p^C.

We start by picking an arbitrary $\phi \in (0, 1)$. Let p^* be the corresponding optimal investment threshold, and f is the corresponding value function. Function f thus satisfies the differential equation (23), which we rewrite with substitution $\lambda = \frac{\gamma}{1-\phi}$:

$$\frac{1}{2}\sigma^2 p^2 f''(p) + \mu p f'(p) - r f(p) - \frac{\gamma}{1-\phi} f(p) + \frac{\gamma\phi}{1-\phi} f\left(\frac{p}{\phi}\right) = 0. \tag{77}$$

In addition, f satisfies:

$$f(p^*) = \frac{p^*}{r-\mu} - 1, \tag{78}$$

$$f'(p^*) = \frac{1}{r-\mu}, \tag{79}$$

$$f\left(\frac{p^*}{\phi}\right) = \frac{p^*}{\phi(r-\mu)} - 1, \tag{80}$$

$$\lim_{p \to 0} f(p) = 0. \tag{81}$$

23

Since (77) holds for all $p \leq p^*$, also the following two equations that are obtained by differentiating it once and twice, respectively, hold when $p < p^*$:

$$0 = \frac{1}{2}\sigma^2 p^2 f^{(3)}(p) + (\sigma^2 + \mu) p f''(p)$$
$$+ \left(\frac{\gamma}{1-\phi} + r - \mu\right) f'(p) + \frac{\gamma}{1-\phi} f'\left(\frac{p}{\phi}\right), \qquad (82)$$

$$0 = \frac{1}{2}\sigma^2 p^2 f^{(4)}(p) + (2\sigma^2 + \mu) p f^{(3)}(p)$$
$$+ (\sigma^2 - r + 2\mu) f''(p) - \frac{\gamma}{1-\phi}\left(f''(p) - \frac{1}{\phi}f''\left(\frac{p}{\phi}\right)\right). \qquad (83)$$

Further, (82) and (83) must also hold at $p = p^*$, when we define $f''(p^*)$, $f^{(3)}(p^*)$, and $f^{(4)}(p^*)$ to be the left-hand side derivatives. Therefore, we adopt the definitions:

$$f^{(i)}(p^*) \equiv \lim_{p \to (p^*)-} f^{(i)}(p), \ i = 2, 3, 4. \qquad (84)$$

Substituting (78)-(80) in (77) and simplifying, we find that at $p = p^*$:

$$f''(p^*) = \frac{p^* - r - \gamma}{\frac{1}{2}\sigma^2 (p^*)^2}. \qquad (85)$$

Similarly, substituting (78)-(80) and (85) in (82), we find that

$$f^{(3)}(p^*) = \frac{(\sigma^2 + \mu) p^* (r + \gamma - p^*) + 1}{\frac{1}{4}\sigma^4 (p^*)^4}. \qquad (86)$$

Finally, substituting (78)-(80), (85), and (86) in (82), we get:

$$f^{(4)}(p^*) = \frac{1}{\frac{1}{2}\sigma^2 (p^*)^2}\left\{-(2\sigma^2 + \mu) p^*\left(\frac{(\sigma^2 + \mu) p^* (r + \gamma - p^*) + 1}{\frac{1}{4}\sigma^4 (p^*)^4}\right)\right.$$
$$\left. - (\sigma^2 - r + 2\mu)\left(\frac{p^* - r - \gamma}{\frac{1}{2}\sigma^2 (p^*)^2}\right) + \frac{\gamma}{1-\phi}\left(\frac{p^* - r - \gamma}{\frac{1}{2}\sigma^2 (p^*)^2}\right)\right\}. \qquad (87)$$

Now, take $\overline{\phi} \in (\phi, 1]$ and consider the same threshold level p^* as before. Denote by $\overline{f}$ a function that satisfies (78)-(80) at $p = p^*$, and (77) for $p \leq p^*$, *but with ϕ replaced by $\overline{\phi}$.*

Since $\overline{f}$ satisfies (78) and (79), we have $\overline{f}(p^*) = f(p^*)$ and $\overline{f}'(p^*) = f'(p^*)$. We can now derive the expressions for $\overline{f}''(p^*)$, $\overline{f}^{(3)}(p^*)$, and $\overline{f}^{(4)}(p^*)$ in the same way as for f. This leads to the same expressions as (85), (86), and (87), but with ϕ replaced by $\overline{\phi}$. Since (85) and (86) do not depend on ϕ, we find that $\overline{f}''(p^*) = f''(p^*)$, and $\overline{f}^{(3)}(p^*) = f^{(3)}(p^*)$. However, (87) depends on ϕ, and it is easy to confirm that $\overline{f}^{(4)}(p^*) > f^{(4)}(p^*)$.

Consider the values of functions f and $\overline{f}$ just below p^*, i.e., fix $p^- = p^* - \varepsilon$, where ε is a very small positive number. When p^- is sufficiently close to p^*, we get the value $f(p^-)$ expanding f about p^*:

$$f(p^-) = f(p^*) + (p^- - p^*) f'(p^*) + \frac{1}{2!}(p^- - p^*)^2 f''(p^*)$$
$$+ \frac{1}{3!}(p^- - p^*)^3 f^{(3)}(p^*) + \frac{1}{4!}(p^- - p^*)^4 f^{(4)}(p^*) + O\left[(p^- - p^*)^5\right] \qquad (88)$$

24

where $O\left[\left(p^- - p^*\right)^5\right]$ represents terms of order 5 and higher. A similar expression can be written for $\overline{f}\left(p^-\right)$. Consider then the difference $\overline{f}\left(p^-\right) - f\left(p^-\right)$. Since $\overline{f}^{(i)}\left(p^*\right) = f^{(i)}\left(p^*\right)$ for $i = 0, 1, 2, 3$, we find that

$$\overline{f}\left(p^-\right) - f\left(p^-\right) = \frac{1}{4!}\left(p^- - p^*\right)^4 \left(\overline{f}^{(4)}\left(p^*\right) - f^{(4)}\left(p^*\right)\right) + O\left[\left(p^- - p^*\right)^5\right]. \quad (89)$$

Since $\overline{f}^{(4)}\left(p^*\right) > f^{(4)}\left(p^*\right)$, we have confirmed that $\overline{f}\left(p^-\right) > f\left(p^-\right)$ when p^- is sufficiently close to p^*. Thus, we know that $\overline{f}\left(p^*\right) = f\left(p^*\right)$, but just below p^*, $\overline{f}\left(p\right) > f\left(p\right)$.

The next step is to show that functions $\overline{f}$ and f do not cross each other anywhere below p^*. Note that at p^*, all the derivatives of $\overline{f}$ and f up to the third derivative are equal, but $\overline{f}^{(4)}\left(p^*\right) > f^{(4)}\left(p^*\right)$. This means that just below p^*, $\overline{f}^{(3)}\left(p\right) < f^{(3)}\left(p\right)$, $\overline{f}''\left(p\right) > f''\left(p\right)$, $\overline{f}'\left(p\right) < f'\left(p\right)$, and $\overline{f}\left(p\right) > f\left(p\right)$. Therefore, if ever $\overline{f}$ and f are going to cross each other below p^*, then moving downwards from p^*, there must be some point $\widetilde{p}$ where $\overline{f}''\left(\widetilde{p}\right) = f''\left(\widetilde{p}\right)$, while $\overline{f}''\left(p\right) > f''\left(p\right)$, $\overline{f}'\left(p\right) < f'\left(p\right)$, and $\overline{f}\left(p\right) > f\left(p\right)$ for all $\widetilde{p} < p < p^*$.

To show that this is not possible, take an arbitrary $\widetilde{p} < p^*$, and assume that $\overline{f}''\left(p\right) > f''\left(p\right)$, $\overline{f}'\left(p\right) < f'\left(p\right)$ and $\overline{f}\left(p\right) > f\left(p\right)$ for all $\widetilde{p} < p < p^*$. Equation (77) must naturally be satisfied at this point both for f and $\overline{f}$:

$$\frac{1}{2}\sigma^2\widetilde{p}^2 f''\left(\widetilde{p}\right) + \mu\widetilde{p}f'\left(p\right) - rf\left(\widetilde{p}\right) - \frac{\gamma}{1-\phi}f\left(\widetilde{p}\right) + \frac{\gamma\phi}{1-\phi}f\left(\frac{\widetilde{p}}{\phi}\right) = 0, \quad (90)$$

$$\frac{1}{2}\sigma^2\widetilde{p}^2\overline{f}''\left(\widetilde{p}\right) + \mu\widetilde{p}\overline{f}'\left(\widetilde{p}\right) - r\overline{f}\left(\widetilde{p}\right) - \frac{\gamma}{1-\overline{\phi}}\overline{f}\left(\widetilde{p}\right) + \frac{\gamma\overline{\phi}}{1-\overline{\phi}}\overline{f}\left(\frac{\widetilde{p}}{\phi}\right) = 0. \quad (91)$$

Subtracting these from each other we get:

$$\overline{f}''\left(\widetilde{p}\right) - f''\left(\widetilde{p}\right) = \frac{\mu\widetilde{p}\left(f'\left(\widetilde{p}\right) - \overline{f}'\left(\widetilde{p}\right)\right) + r\left(\overline{f}\left(\widetilde{p}\right) - f\left(\widetilde{p}\right)\right)}{\frac{1}{2}\sigma^2\widetilde{p}^2} +$$
$$\frac{\left[\frac{\gamma}{1-\overline{\phi}}\overline{f}\left(\widetilde{p}\right) - \frac{\gamma\overline{\phi}}{1-\overline{\phi}}\overline{f}\left(\frac{\widetilde{p}}{\phi}\right) - \frac{\gamma}{1-\phi}f\left(\widetilde{p}\right) + \frac{\gamma\phi}{1-\phi}f\left(\frac{\widetilde{p}}{\phi}\right)\right]}{\frac{1}{2}\sigma^2\widetilde{p}^2}. \quad (92)$$

The first term of (92) is clearly positive. Denote the term in brackets by W. It can be written as:

$$W = \frac{\gamma\left[\left(1-\phi\right)\overline{f}\left(\widetilde{p}\right) - \left(1-\phi\right)\overline{\phi}\overline{f}\left(\frac{\widetilde{p}}{\phi}\right) - \left(1-\overline{\phi}\right)f\left(\widetilde{p}\right) + \left(1-\overline{\phi}\right)\phi f\left(\frac{\widetilde{p}}{\phi}\right)\right]}{\left(1-\overline{\phi}\right)\left(1-\phi\right)}. \quad (93)$$

Consider then the term $\phi f\left(\frac{\widetilde{p}}{\phi}\right)$. This can be written as:

$$\phi f\left(\frac{\widetilde{p}}{\phi}\right) = \phi\left[f\left(\widetilde{p}\right) + \left(\frac{\widetilde{p}}{\phi} - \widetilde{p}\right)f'\left(\xi\right)\right] = \phi f\left(\widetilde{p}\right) + \left(1-\phi\right)\widetilde{p}f'\left(\xi\right) \quad (94)$$

for some $\xi \in \left(\widetilde{p}, \frac{\widetilde{p}}{\phi}\right)$. Similarly, we can write

$$\overline{\phi}f\left(\frac{\widetilde{p}}{\phi}\right) = \overline{\phi}f\left(\widetilde{p}\right) + \left(1-\overline{\phi}\right)\widetilde{p}\overline{f}'\left(\overline{\xi}\right) \quad (95)$$

for some $\overline{\xi} \in \left(\widetilde{p}, \frac{\widetilde{p}}{\phi}\right)$. Because $\overline{f}'(p) < f'(p)$ for all $\widetilde{p} < p < p^*$ and $\overline{f}'(p) = f'(p)$ for $p \geq p^*$, it must be that $f'(\xi) > \overline{f}'(\overline{\xi})$. Substituting (94) and (95) in (93) we get:

$$
\begin{aligned}
W &= \frac{\gamma}{\left(1-\overline{\phi}\right)(1-\phi)}\left[(1-\phi)\left(\overline{f}(\widetilde{p}) - \overline{\phi f}(\widetilde{p}) - \left(1-\overline{\phi}\right)\widetilde{p}\overline{f}'(\overline{\xi})\right)\right.\\
&\quad \left. - \left(1-\overline{\phi}\right)\left(f(\widetilde{p}) - \phi f(\widetilde{p}) - (1-\phi)\widetilde{p}f'(\xi)\right)\right]\\
&= \frac{\gamma\left[(1-\phi)\left(1-\overline{\phi}\right)\left(\overline{f}(\widetilde{p}) - \widetilde{p}\overline{f}'(\overline{\xi})\right) - (1-\phi)\left(1-\overline{\phi}\right)\left(f(\widetilde{p}) - \widetilde{p}f'(\xi)\right)\right]}{\left(1-\overline{\phi}\right)(1-\phi)}\\
&= \gamma\left[\overline{f}(\widetilde{p}) - f(\widetilde{p}) + \widetilde{p}\left(f'(\xi) - \overline{f}'(\overline{\xi})\right)\right].
\end{aligned}
\tag{96}
$$

Since $\overline{f}(\widetilde{p}) > f(\widetilde{p})$ and $f'(\xi) > \overline{f}'(\overline{\xi})$, we have $W > 0$, and thus from (92) we get that $\overline{f}''(\widetilde{p}) > f''(\widetilde{p})$. This means that there can not be such $\widetilde{p} < p^*$ where the second derivatives cross each other given that $\overline{f}'(p) < f'(p)$, and $\overline{f}(p) > f(p)$ for all $\widetilde{p} < p < p^*$. This means that $\overline{f}$ can not cross f below p^*, and since $\overline{f}(p) > f(p)$ with p sufficiently close to p^*, it must be that $\overline{f}(p) > f(p)$ everywhere below p^*.

We defined $\overline{f}(p)$ as a function that satisfies (78)-(80) at $p = p^*$, and (77) for $p < p^*$, *but with ϕ replaced by $\overline{\phi}$*. Since all these conditions are satisfied, the level $p = p^*$ would be the optimal investment threshold with $\overline{\phi}$ if also the condition (81) were satisfied. However, we have just shown that $\overline{f}(p) > f(p)$ everywhere below p^*, and moreover, the derivatives up to the second derivative are such that they "move $\overline{f}(p)$ and $f(p)$" away from each other. This means that the condition (81) can not be satisfied by $\overline{f}$. Instead, we have

$$
\lim_{p \to 0}\overline{f}(p) > 0.
\tag{97}
$$

This means that p^* would be the correct investment threshold with $\overline{\phi}$ if the investment option entailed a positive payoff in the case where price would be absorbed to zero. Since in reality the value of the option should be zero in that case, $\overline{f}(p)$ is a too optimistic value for the investment option. In reality the value of the option is lower, which means that it is optimal to give it up in return of the project earlier than at p^*. The correct investment threshold $\overline{p}^*$ with $\overline{\phi} > \phi$ is therefore $\overline{p}^* < p^*$.

Thus, we have shown that the greater the value of ϕ, the lower the investment threshold (while keeping γ fixed). $\blacksquare$

References

[1] Alvarez, L., Stenbacka, R., 2001. Adoption of Uncertain Multi-stage Technology Projects: A Real Options Approach. Journal of Mathematical Economics 35, 71-97.

[2] Alvarez, L., Stenbacka, R., 2002. Strategic Adoption of Intermediate Technologies: A Real Options Approach. The Yrjö Jahnsson Working Paper Series in Industrial Economics 4 (2002), Swedish School of Economics and Business Administration, Helsinki.

[3] Amundsen, E., Mortensen, J., 2001. The Danish Green Certificate System: Some Simple Analytical Results. Energy Economics 23, 489-509.

[4] Balcer, Y., Lippman, S., 1984. Technological Expectations and Adoption of Improved Technology. Journal of Economic Theory 34, 292-318.

[5] Brekke, K., Schieldrop, B., 1999. Investment in Flexible Technologies under Uncertainty, in: Brennan, M.J., Trigeorgis, L. (Eds.), Project Flexibility, Agency, and Competition: New Developments in the Theory of Real Options, Oxford University Press, New York.

[6] Brennan, M., Schwartz, E, 1985. Evaluating Natural Resource Investments. Journal of Business 58, 135-157.

[7] Dixit, A.K., Pindyck, R.S., 1994. Investment under Uncertainty. Princeton University Press, Princeton, New Jersey.

[8] European Commission, 1997. Energy for the Future: Renewable Sources of Energy. Communication for the Commission, White Paper for a Community Strategy and Action Plan, COM(97).

[9] European Commission, 2001. Towards a European Strategy for the Security of Energy Supply. Green Paper, Technical Background Document, Luxembourg: Office for Official Publications of the European Communities.

[10] Farzin, Y., Huisman, K., Kort, P., 1998. Optimal Timing of Technology Adoption. Journal of Economic Dynamics and Control 22, 779-799.

[11] Fudenberg, D., Tirole, J., 1985. Preemption and Rent Equalization in the Adoption of New Technology. Review of Economic Studies 52, 383-401.

[12] Grenadier, S., Weiss, A., 1997. Investment in Technological Innovations: An Option Pricing Approach. Journal of Financial Economics 44, 397-416.

[13] Hoppe, H., 2000. Second-Mover Advantages in the Strategic Adoption of New Technology under Uncertainty. International Journal of Industrial Organization 18, 315-338.

[14] Hunt, S., Shuttleworth, G., 1996. Competition and Choice in Electricity. John Wiley & Sons Ltd, Chichester.

[15] Jensen, S., Skytte, K., 2002. Interactions between the Power and Green Certificate Markets. Energy Policy 30, 425-435.

[16] Karatzas, I., Shreve, S.E., 1998. Methods of Mathematical Finance. Applications of Mathematics, Stochastic Modelling and Applied Probability 39. Springer-Verlag, New York.

[17] Krohn, S., 2002. Danish Wind Turbines: An Industrial Success Story. Available at the www-site of the Danish Wind Industry Association: www.windpower.org

[18] Lumley, R., Zervos, M., 2001. A Model for Investments in the Natural Resource Industry with Switching Costs. Mathematics of Operations Research 26, 637-653.

[19] Martzoukos, S., Teplitz-Sembitzky, W., 1992. Optimal Timing of Transmission Line Investments in the Face of Uncertain Demand. Energy Economics 14, 3-10.

[20] McDonald, R., Siegel, D., 1986. The Value of Waiting to Invest. Quarterly Journal of Economics 101, 707-728.

[21] Øksendal, B., 2000. Stochastic Differential Equations: An Introduction with Applications. 5th edition (second printing), Springer, Berlin.

[22] Paddock, J., Siegel, D., Smith, J., 1988. Option Valuation of Claims on Real Assets: The Case of Offshore Petroleum Leases. Quarterly Journal of Economics 103, 479-508.

[23] Pindyck, R., 1993. Investments of Uncertain Cost. Journal of Financial Economics 34, 53-76.

[24] Pindyck, R., 1988. Irreversible Investment, Capacity Choice, and the Value of the Firm. American Economic Review 78, 969-985.

[25] Rahman, A., Loulou, R., 2001. Technology Acquisition with Technological Progress: Effects of Expectations, Rivalry and Uncertainty. European Journal of Operational Research 129, 159-185.

[26] Reinganum, J., 1981. On the Diffusion of New Technology: A Game Theoretic Approach. Review of Economic Studies 153, 395-406.

[27] Venetsanos, K., Angelopoulou, P., Tsoutsos, T., 2002. Renewable Energy Sources Project Appraisal under Uncertainty: The Case of Wind Energy Exploitation within a Changing Energy Market Development. Energy Policy 30, 293-307.