Spatio–Temporal Chaos in the Transverse Section of Gyrotron Resonators

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Abstract—**Nonstationary oscillations in gyrotrons are investigated for large azimuthal indices of modes leading in the transverse section of gyrotron resonators to spatial chaos in azimuthal direction. Limiting values of the azimuthal index m of the mode, beyond which stationary single-mode operation of a gyrotron becomes impossible, are found.**

Index Terms—**Gyrotron, spatio–temporal chaos.**

I. INTRODUCTION

IN CONNECTION with striving to develop high-frequency high-power gyrotrons needed for modern fusion reactors, it is appropriate to study under what circumstances chaos-like processes can occur in gyrotrons. Recently [1], [2], we studied the onset of stochastic oscillations in gyrotrons by means of the self-consistent theory describing nonstationary processes. Complicated alternating sequences of regions of stationary, automodulation, and chaotic oscillations were found in the plane of the generalized gyrotron variables: cyclotron resonance mismatch and dimensionless current. Traces of period doubling and intermittency on the route to chaos were discovered. In those studies, it was assumed that the transverse structure of RF field in the resonator is fixed and can be represented by a single TE mode of a cylindrical resonator. If the diameter of a resonator is much larger than the wavelength, as is the case in high-frequency and high-power gyrotrons, many resonator modes may fall within the cyclotron resonance band. In such a case, as was suggested in [3], it is meaningful to go over from a single-mode representation of RF field to a spatio–temporal description in which the envelope of the RF field is a continuous function of transversal coordinates. In [4], this approach was generalized to the case of variable azimuthal as well as longitudinal coordinates. In the present paper, we present a simple and mathematically transparent method of solving the underlying equations. Detailed calculations result in specifying critical values of the azimuthal index of the mode beyond which gyrotron oscillations become irregular in the azimuthal direction at particular values of cyclotron resonance mismatch and dimensionless current.

II. GYROTRON EQUATIONS

Electron motion and the time- and spatially dependent high-frequency field in the resonator can be calculated from the following [4]:

$$\begin{cases} \dfrac{dp}{d\zeta} + i(\Delta + |p|^2 - 1)p = if(\zeta, \xi, \tau) \\ \dfrac{\partial^2 f}{\partial \zeta^2} - i\dfrac{\partial f}{\partial \tau} - i\dfrac{\partial f}{\partial \xi} + \delta f = \dfrac{I}{2\pi}\int_0^{2\pi} p\,d\vartheta_0 \end{cases} \quad (1)$$

where p is the complex transverse momentum of the electron normalized to its initial absolute value, $\zeta = (\beta_{\perp 0}^2 \omega / 2\beta_{\|0} c)z$ and $\xi = (1/8)\alpha^2\beta_{\perp 0}^2 m\varphi$ are dimensionless axial and azimuthal coordinates, respectively, $\beta_{\perp 0} = v_{\perp 0}/c$ and $\beta_{\|0} = v_{\|0}/c$ are normalized electron velocities, $\alpha = \beta_{\perp 0}/\beta_{\|0}$ is the pitch factor, $\Delta = 2(\omega - \omega_c)/\beta_{\perp 0}^2\omega$ is the frequency mismatch, $\omega_c/2\pi = 28B/\gamma_{\text{rel}}$ is the electron cyclotron frequency in GHz, B is the magnetic field in T, γ_{rel} is the relativistic factor of electrons, $f(\zeta, \xi, \tau)$ is the high-frequency field in the resonator, $\tau = (1/8)\beta_{\perp 0}^4\beta_{\|0}^{-2}\omega_c t$ is the dimensionless time, $\delta = 8\beta_{\|0}^2\beta_{\perp 0}^{-4}[\overline{\omega} - \omega(\zeta)]\omega_c^{-1}$ describes variation of the cut-off frequency $\omega(\zeta)$ along the resonator axis, $\overline{\omega}$ is the cut-off frequency at the exit from the resonator, and I is the dimensionless current

$$I = 9.4 \cdot 10^{-4} I_0 \beta_{\|0} \beta_{\perp 0}^{-6} \frac{J_{m\pm 1}^2\left(\frac{2\pi}{\lambda}R_{\text{el}}\right)}{\gamma_{\text{rel}}(\nu^2 - m^2)J_m^2(\nu)}. \quad (2)$$

Here, I_0 is the beam current in amperes, J is the Bessel function, m is the azimuthal index of the mode, λ is the wavelength, R_{el} is the electron beam radius, and ν is the zero of the derivative of the Bessel function. This description is valid for operation at the fundamental cyclotron resonance.

The system of (1) has to be supplemented by the standard initial condition for the momentum, $p(0) = \exp(i\vartheta_0)$ with $0 \leq \vartheta_0 \leq 2\pi$, and by the boundary condition for the field at the entrance to the interaction space

$$f(0, \tau) = 0 \quad (3)$$

which means that at the entrance the field must vanish. At the exit from the interaction space ($\zeta = \zeta_{\text{out}}$) the so-called reflectionless boundary condition is applied

$$\left(f(\zeta, \xi, \tau) + \frac{1}{\sqrt{\pi i}}\int_0^\tau \frac{1}{\sqrt{\tau - \tau'}}\frac{\partial f(\zeta, \xi, \tau')}{\partial \zeta}\,d\tau' \right)\Bigg|_{\zeta=\zeta_{\text{out}}} = 0. \quad (4)$$

In the azimuthal direction, periodic boundary conditions are used.

III. Numerical Method

There is no need for introduction of the special variable $u = \tau - \xi$ in (1), as was done in [4]. This restricts the class of solutions and complicates the matter. Equation (1) can be solved directly numerically by generalizing the method used in [1] into two spatial dimensions. We discretize the domain of interest with step sizes $\Delta\zeta$, $\Delta\xi$, and $\Delta\tau$. The values of f at grid points are denoted as

$$f(j\Delta\zeta, k\Delta\xi, n\Delta\tau) \equiv f_{j,k}^n,$$

where $j = 0, 1, \ldots, J$; $k = 0, 1, \ldots, K - 1$; and $n = 0, 1, \ldots$. Using this notation, we can replace the derivatives in (1) by the following approximations:

$$\frac{\partial^2 f}{\partial\zeta^2} \longrightarrow \frac{f_{j+1,k}^{n+1} - 2f_{j,k}^{n+1} + f_{j-1,k}^{n+1}}{(\Delta\zeta)^2} \tag{5}$$

$$\frac{\partial f}{\partial\tau} \longrightarrow \frac{f_{j,k}^{n+1} - f_{j,k}^n}{\Delta\tau} \tag{6}$$

$$\frac{\partial f}{\partial\xi} \longrightarrow \frac{f_{j,k+1}^{n+1} - f_{j,k-1}^{n+1}}{2\Delta\xi}. \tag{7}$$

This discretization scheme is centered in both spatial dimensions. The current term is replaced by

$$g_{j,k}^n \equiv \frac{I}{s} \sum_{l=1}^s p\left(j\Delta\zeta, k\Delta\xi, n\Delta\tau, \vartheta_0^{(l)}\right). \tag{8}$$

As far as the boundary condition (4) is concerned, we can isolate the singularity in it if the function $\partial f(\zeta, \xi, \tau')/\partial\zeta$ satisfies the Lipschitz condition. As a result, we obtain a difference approximation for (4)

$$f_{J,k}^n + \sqrt{\frac{\Delta\tau}{\pi i}}\left(2\sqrt{n}\frac{f_{J,k}^n - f_{J-1,k}^n}{\Delta\zeta}\right.$$
$$\left. + \sum_{\nu=0}^{n-1}\frac{f_{J,k}^\nu - f_{J-1,k}^\nu - f_{J,k}^n + f_{J-1,k}^n}{\Delta\zeta\sqrt{(n-\nu)}}\right) = 0 \tag{9}$$

for $k = 0, 1, \ldots, K - 1$.

The discrete problem can now be written in the following matrix form:

$$A^n \mathbf{f}^{n+1} = \mathbf{c}^n \tag{10}$$

where

$$\mathbf{f}^n = (\, f_{0,0}^n \quad \cdots \quad f_{0,K-1}^n \quad f_{1,0}^n \quad \cdots \quad f_{J,K-1}^n \,)^{\mathrm{T}}$$
$$\mathbf{c}^n = (\, c_{0,0}^n \quad \cdots \quad c_{0,K-1}^n \quad c_{1,0}^n \quad \cdots \quad c_{J,K-1}^n \,)^{\mathrm{T}}$$

with

$$c_{0,k}^n = 0$$
$$c_{j,k}^n = g_{j,k}^n - \frac{i}{\Delta\tau}f_{j,k}^n \qquad \text{for } 1 \le j \le J - 1,$$
$$c_{J,k}^n = \sum_{\nu=0}^n \frac{f_{J-1,k}^\nu - f_{J,k}^\nu}{\sqrt{n+1-\nu}}.$$

The coefficient matrix has a block tridiagonal form

$$A^n = \begin{pmatrix} I & & & & & \\ a_1 I & B & a_1 I & & & \\ & a_1 I & B & a_1 I & & \\ & & \ddots & \ddots & \ddots & \\ & & & a_1 I & B & a_1 I \\ & & & & a_2^n I & a_3^n I \end{pmatrix}$$

where I is the $K \times K$ identity matrix

$$B = \begin{pmatrix} b_1 & -b_2 & & & b_2 \\ b_2 & b_1 & -b_2 & & \\ & \ddots & \ddots & \ddots & \\ & & b_2 & b_1 & -b_2 \\ -b_2 & & & b_2 & b_1 \end{pmatrix}$$

is a cyclic tridiagonal matrix due to the periodic boundary condition in ξ, and

$$a_1 = \frac{1}{(\Delta\zeta)^2}$$
$$a_2^n = -2\sqrt{n+1} + \sum_{\nu=0}^n \frac{1}{\sqrt{n+1-\nu}}$$
$$a_3^n = \Delta\zeta\sqrt{\frac{\pi i}{\Delta\tau}} + 2\sqrt{n+1} - \sum_{\nu=0}^n \frac{1}{\sqrt{n+1-\nu}}$$
$$b_1 = -\frac{2}{(\Delta\zeta)^2} - \frac{i}{\Delta\tau}$$
$$b_2 = \frac{1}{2\Delta\xi}.$$

The first equation in (1) is solved by a standard Runge–Kutta algorithm using the already known field $f(\zeta, \xi, \tau)$ to obtain the discrete current term.

IV. Computations

The effect of the azimuthal mode number m on the stability of gyrotron oscillations was studied by solving the system (10) for different values of m and finding the lowest one at which stationary oscillations break down. Many combinations of Δ and I were used in the computations—they were chosen to cover the region marked "stationary" in [1, Fig. 1].

We started all computations with an initial condition

$$f(\zeta, \xi, 0) = \left[0.1 + 0.01\sin\left(\frac{2\pi\xi}{\xi_{\max}}\right)\right]\sin\left(\frac{\pi\zeta}{\zeta_{\text{out}}}\right) \tag{11}$$

which simulates a field profile with only one maximum in the axial direction and a small perturbation around the azimuth. Here, $\xi_{\max} = (\pi/4)\alpha^2\beta_{\perp 0}^2 m$.

In the calculations, we have assumed a cylindrical resonator $[\delta(\zeta) \equiv 0]$ with $\zeta_{\text{out}} = 15$, fixed the values $\beta_{\perp 0} = 0.426$ and $\beta_{\parallel 0} = 0.316$ for velocities corresponding to $\alpha = 1.35$ and the accelerating voltage $U = 92$ kV, and used a 61×40 or 61×60 spatial grid depending on m. The latter one leads to a problem of 3660 unknowns and, consequently, the coefficient

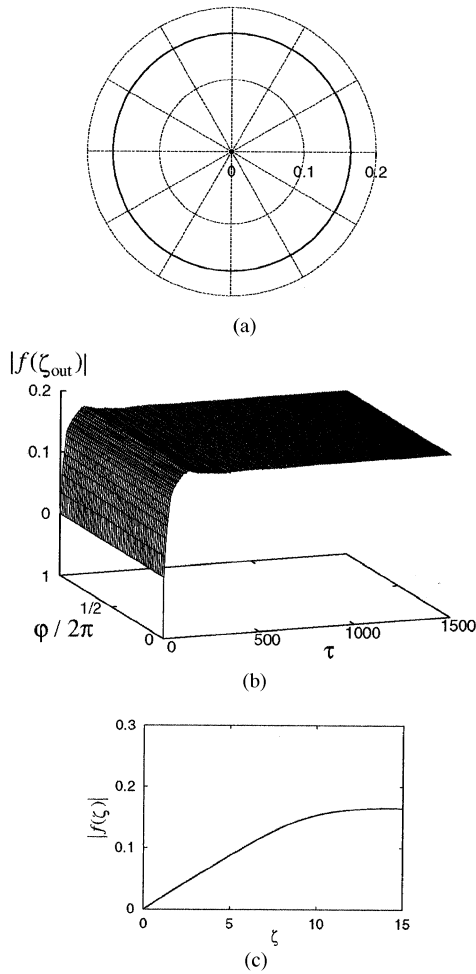Fig. 1. $I = 0.01$, $\Delta = 0.60$, and $m = 45$. (a) Stationary RF field amplitude at the output cross section as a function of azimuthal angle. (b) RF field amplitude at the output cross section as a function of azimuthal angle and time. (c) Axial profile of the stationary RF field.
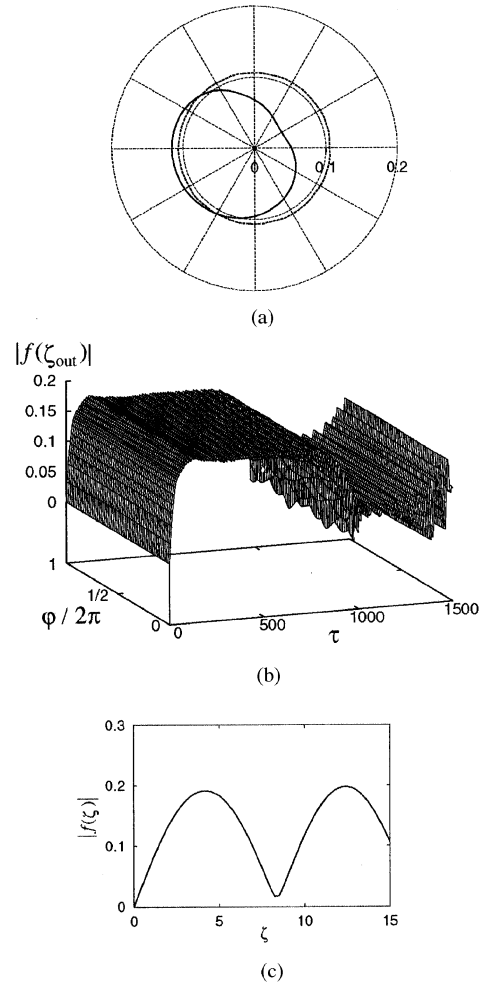

Fig. 2. $I = 0.01$, $\Delta = 0.60$, and $m = 50$. (a) RF field amplitude at the output cross section as a function of azimuthal angle at $\tau = 950$ (solid line) and at $\tau = 4000$ (dashed line). (b) RF field amplitude as a function of azimuthal angle and time. (c) Axial profile of the RF field at $\tau = 4000$.

matrix has about 10^7 elements. Since only about 10^4 of these are nonzero, the solution can be efficiently found using sparse matrix techniques. We applied the IMSL routine `DLSLZG` [5] and were able to proceed with small temporal steps ($\Delta\tau = 0.1$) at a reasonable rate.

V. RESULTS

Our extensive study of the solutions for (1) in the parameter space reveals that essentially three types of oscillations exist. Starting with a small m in the region of stationary oscillations, we can expect practically the same kind of axial field profile and temporal behavior as was seen in our previous study [1] in only one spatial dimension. However, with increasing m, we always encounter a limit where the stationary solution breaks down. We denote by m_{crit} the highest m yielding a stationary field.

Stationary oscillations are illustrated by Fig. 1, which corresponds to the operating parameters $I = 0.01$, $\Delta = 0.60$, and $m = 45$. In Fig. 1(a), we show the RF field amplitude at $\zeta = \zeta_{\mathrm{out}}$ as a function of azimuthal angle. In Fig. 1(b), the same quantity is plotted as a function of the angle and time. Fig. 1(c) shows the axial field profile.

Moving to a slightly higher m while staying at the same point in the (Δ, I) plane, we find a dramatic difference. As is shown in Fig. 2, with $m = 50$ there is a transition period during which the field changes from the optimal shape shown in Fig. 1(c) into a less favorable configuration with two maxima in the axial direction, shown in Fig. 2(c). During this transition caused by competition of modes with different axial indices, the field structure varies rapidly in time and space and resembles transient chaos.

To illustrate spatio–temporal chaos, we show similar figures for $I = 0.10$, $\Delta = 0.20$, and $m = 10$. For such a high current, the field soon becomes very irregular both in axial and azimuthal coordinates (see Fig. 3).

The values of m_{crit} for different I and Δ are shown in Fig. 4. At some points in the (Δ, I) plane, the difference between stationary and nonstationary oscillations shows up too slowly for practical detection. We estimate the error of the given values of m_{crit} to be ± 2.

VI. CONCLUSION

Our study shows that there exist maximal values m_{crit} of the azimuthal index m beyond which stationary single-mode op-
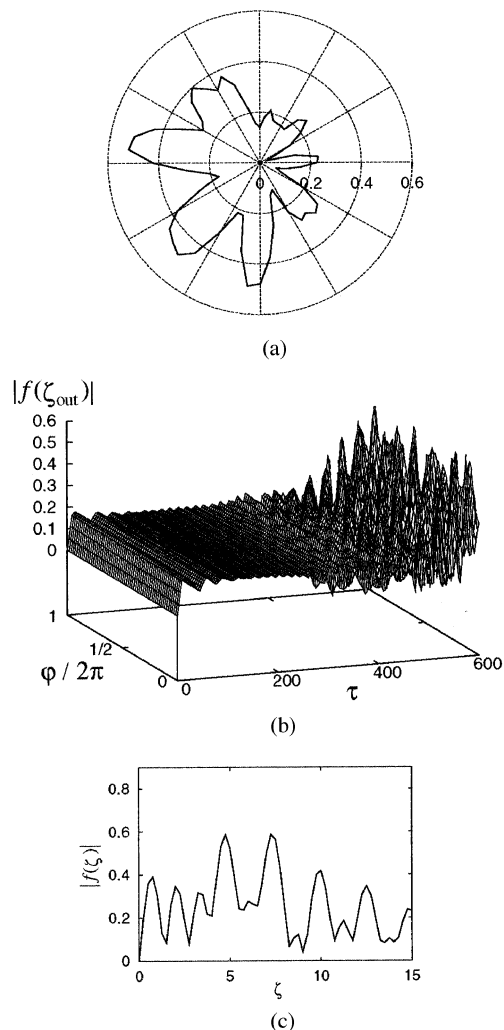
(a)

(b)

(c)

Fig. 3. $I = 0.10, \Delta = 0.20, m = 10$. (a) Momentary RF field amplitude at the output cross section as a function of azimuthal angle at $\tau = 570$. (b) RF field amplitude as a function of azimuthal angle and time. (c) Axial profile of the RF field at $\tau = 570$ and $\varphi = 0$.

eration of a gyrotron is no longer possible. Surprisingly, these critical numbers are rather low ($m_{\text{crit}} \lesssim 46$) for a high-power gyrotron with typical operation parameters $\alpha = 1.35$ and $U = 92$ kV (compare with [4]) even in the region of highest efficiencies. This puts natural limits for the attempts to minimize Ohmic losses by going over to large-diameter cavities and to high-order whispering gallery operating modes. This means also that the azimuthal index m of the mode of a gyrotron which could operate both in the fundamental and second harmonic should be $\lesssim 23$).

VII. Outlook

The onset of stochastic oscillations has also been studied experimentally [6], [7], in the case of low-order modes, low currents, tapered waveguide, and reflections. It has been found that stochastic oscillations already appear at a few amperes. For example, the stochastic operation point of the gyrotron with a tapered wavequide and reflections studied in [7] corresponds to
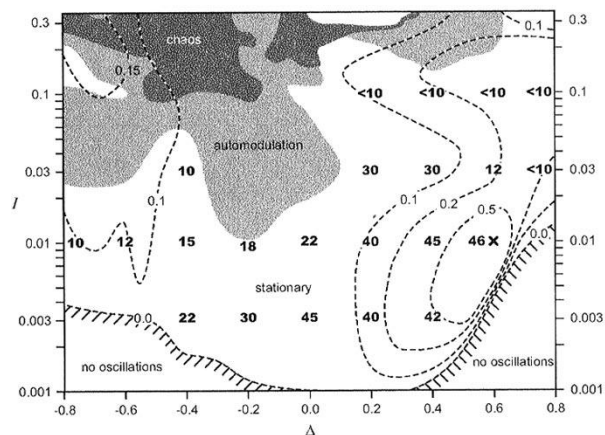

Fig. 4. The upper limit in m for stationary gyrotron oscillations in the (Δ, I) plane (numbers in boldface). The background shows the topology of different kinds of oscillations of a gyrotron according to one-dimensional calculations [1]. Here, white regions correspond to stationary oscillations, gray regions correspond to automodulation, and dark regions to chaotic oscillations. The contours of constant efficiency are shown by the dashed curves, and the point of maximum efficiency $\eta_\perp^{\max} = 0.75$ is marked by a cross.

$\Delta = 0.75$ and $I = 0.02$. However, this is not in conflict with our results presented in Fig. 4. The point is that in the present study we restrict ourselves to cylindrical cavities [$\delta = 0$ in (1)] and consider only the reflectionless case [(4)]. It can be expected that reflections [6], and tapering [8], [9] significantly lower the border of onset of stochastic oscillations. Generalized theory with allowance for reflections will be published somewhere else [10].

Finally, we would like to mention the subject of controlling chaos which has not yet been studied in gyrotrons. For example, one could imagine to control chaos by converting a chaotic attractor to any one of a large number of possible attracting time-periodic motions by making only small time-dependent perturbations of an available system parameter [11]. In the case of gyrotrons for $m < m_{\text{crit}}$ in the regions where only temporal chaos occurs, one would wish to stabilize definite shapes of $|f(\zeta)|$, e.g., those with only one or two maxima which provide high-gyrotron efficiency. Is this feasible by making only *small* time-dependent adjustments of control parameters Δ and I? Probably yes, if done quickly enough. A study of this possibility is beyond the scope of the present paper.

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